Repeated Games with Observable Actions in Continuous Time: Costly Transfers in Repeated Cooperation*

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August 2, 2019

Abstract

I propose a way to formulate and solve for subgame perfect equilibria of continuous-time repeated games with both observable and unobservable actions. The main idea is that instead of first defining an extensive-form game in continuous time, one can look directly for self-enforcing agreements corresponding to the strategic interaction at hand. To discipline players' observable deviations, I impose an inertia restriction that makes the deviator stuck with his observable action for a small amount of time. This restriction simultaneously preserves the tractability of the model, and ensures that agreements and deviating strategies are well defined.

To illustrate this idea, I consider an example of two cartel members colluding in a continuous-time repeated setting with imperfectly observable productive actions and observable money transfers. Money transfers are costly: only a fraction $k < 1$ of the money sent is received by the recipient (with the case $k = 0$ corresponding to pure money burning). I introduce the notion of a self-enforcing public agreement which mimics the notion of a pure-strategy public perfect equilibrium from discrete time. For a fixed interest rate $r > 0$, I characterize the set of payoffs attainable in self-enforcing public agreements, as well as the dynamics in optimal ones. Adding the possibility of costly transfers increases the set of attainable payoffs because it allows the promised continuation values to reflect away from the players' individual rationality constraints. In optimal agreements, costly transfers are used rarely and only after extreme histories when the individual rationality constraint of one of the players binds.

*I am especially indebted to Sergey Vorontsov, who contributed greatly to our discussion at the early stage of the project. I am extremely grateful to Michael Ostrovsky, Andrzej Skrzypacz, and Robert Wilson for their extensive guidance. I am also thankful to Dilip Abreu, Benjamin Brooks, Sebastian Di Tella, Elliot Lipnowski, Erik Madsen, Yuliy Sannikov, Takuo Sugaya, and Alexander Wolitzky for their thoughtful comments and suggestions. I also thank seminar participants at Higher School of Economics, MIT/Harvard, New Economic School, Penn State, Stanford, University of New South Wales, University of Wisconsin Madison, University of Melbourne, University of Queensland, AEA 2017, ESAM 2019, Stony Brook Summer Festival 2019.

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1 Introduction

Continuous-time models have received extensive attention from economic theorists in the last two decades. The tractability achieved by studying strategic interactions in continuous time can hardly be overstated. Sannikov (2008) inspired the literature on continuous-time contracts. In another paper, Sannikov (2007) formulated repeated games with imperfectly observable actions in continuous time and developed techniques for finding their pure-strategy public perfect equilibria (p-PPEs). Yet little progress has been made toward rigorously addressing continuous-time games with observable actions. Simon and Stinchcombe (1989) pointed out some of the potential problems.

This paper is methodological. I propose a model of a repeated game in continuous time with *both* perfectly and imperfectly observable actions, and solve for its p-PPEs. Though I consider only one specific example, I believe the main idea of the paper can be used quite generally for finding subgame perfect Nash equilibria of continuous-time games with observable actions. In a companion paper (Panov (2019)), I discuss this idea in more detail and illustrate how to apply it in a series of examples from the existing literature.

The main idea of this paper can be described as follows. Consider first the problem of finding subgame perfect equilibria (SPNEs) for a given strategic interaction in discrete time. There exist two different methods for doing this. The standard approach is described in the following two steps:

1. Represent the strategic interaction as an extensive-form game: define the players' strategies, the outcomes induced by each strategy profile, and the payoffs delivered to the players in each outcome.

2. For the constructed game, compute all Nash equilibria that satisfy subgame perfection.

The second method was proposed by Abreu (1988). I call it *the Abreu approach.* Essentially, the Abreu approach reverses the order of the steps is the standard approach as follows:

1. Consider an *agreement* that is a collection of an initial outcome and punishment outcomes. The initial outcome specifies the whole path of play from the beginning, assuming that nobody makes an observable deviation. For any finite sequence of observed deviations, the corresponding punishment outcome specifies the continuation path of play, assuming no further observable deviations.

2. Given an agreement, define strategies for each player relative to the agreement. A strategy specifies for each outcome the sequence of unobservable actions that may depend on the player's history, as well as the rule of when and how to observably deviate from the outcome. Define the payoff from each strategy relative to the agreement. Call an agreement *self-enforcing* if there is no strategy for any player that constitutes a profitable deviation after some history of play. Finally, find SPNEs by finding all self-enforcing agreements.

For discrete-time interactions, the two approaches lead to the same answer. Yet the Abreu approach is often more tractable (for example, in the case of infinitely repeated games).

For continuous-time interactions with observable actions, following the standard approach is problematic. Indeed, a considerable difficulty appears already at the beginning of the first step: in continuous time, well defined extensive-form strategies for the players may not determine uniquely the corresponding outcome.[1] The main insight of this paper is that the Abreu approach still works well for continuous-time models with observable actions. In other words:

> `The Main Idea:` To find subgame perfect equilibria of strategic interactions with observable actions in continuous time, one can use the Abreu approach. That is, rather than first defining the whole extensive-form game, one can search directly for self-enforcing agreements corresponding to the interaction.

To illustrate how this idea can be used to deal with continuous-time repeated games with observable actions, I consider the following economic example. Two players collude in a continuous-time repeated setting. At each point in time, they can choose productive actions. These actions are imperfectly observable by their effect on the drift of a public Brownian signal. Besides hidden productive actions, the players are allowed to transfer money to each other. These transfers are instantaneously and perfectly observable. Money transfers are costly: there is an exogenous retention parameter $k \in [0, 1)$. If at time t, a player sends the opponent γ amount of money, the opponent immediately receives only $k\gamma$, with the remaining $(1-k)\gamma$ being permanently lost. The limiting case $k = 1$ corresponds to perfect transfers. The motivation to study costly transfers is that in cartels, perfect transfers may often be infeasible (e.g., legally prohibited). The case $k = 0$ corresponds to pure money burning. In cartels, money burning can be implemented via open charity donations, for example, or via any other expenditures which are not directly beneficial to the stockholders of the interacting firms. The intermediate case $k \in (0, 1)$ may be implemented, for instance, when a firm or its subsidiary buys the final product from the competitor or its subsidiary (see Harrington and Skrzypacz (2007)).

The question then is how the possibility of costly transfers may further help the players to sustain cooperation. Note that this case is qualitatively quite different from the case of perfect transfers. When transfers are perfect, one can see already in discrete time that optimal cooperation can be implemented via stationary equilibria (e.g., Levin (2003), Goldlücke and Kranz (2012)). When transfers are costly, this result no longer holds. Indeed, the losses associated with transfers introduce an additional trade-off between providing incentives via transfers today and postponing the costs of transfers into the future. Thus, it is not optimal to use costly transfers regularly at the

[1] For instance, consider a one-player situation in which at each time $t \in [0, \infty)$, the player chooses an action $a_t \in [0, 1]$. Now consider the following strategy which recommends an action to the player as a function of the history of play. At $t = 0$, choose $a_0 = 0$. For all $t > 0$, choose $a_t = \sup_{s \in [0,t)} a_s$. Note that this strategy uniquely determines what the player should choose after any history of his play. Yet it does not uniquely determine the outcome. Indeed, any weakly increasing continuous path of actions a_t with $a_0 = 0$ would fit the description of this strategy.

end of each period. Also, as this trade-off breaks the stationarity of optimal cooperation, solving the model in closed form in discrete time does not seem tractable.

For this continuous-time setting, I study *self-enforcing public agreements* that correspond to p-PPEs of discrete-time games. An agreement is called *self-enforcing*, if there is no strategy for any player that constitutes a strictly profitable deviation after some history of play. To discipline observable deviations, I impose a certain *inertia restriction*. Intuitively, an agreement satisfies the inertia with parameter $\epsilon > 0$ if after an observed deviation, the deviator is stuck with his deviating action for ϵ amount of time. This restriction does not seem too severe. For instance, it is automatically satisfied in any discrete-time model. The exact formulation of the inertia restriction used in this paper is slightly different from the above for tractability reasons. Yet I believe it captures the above intuition. Note that such a inertia allows one to simultaneously incorporate two attractive properties into the model. First, it puts no constraints on the initial path of play. Thus, it permits solving the model in closed form. Second, it imposes regularity on the structure of deviations, which is needed for the one-stage deviation principle to apply. The inertia makes it costly to observably deviate since the deviator suffers a loss in flexibility. Absent any such loss, it is not clear what would prevent players from deviating arbitrarily often, which would render agreements ill-defined.

To find the set of payoffs attainable in self-enforcing public agreements as well as the dynamics in optimal ones, I follow a three-step procedure, which is similar to the cookbook procedure for determining p-PPEs of repeated games in discrete time:

1. Derive the appropriate Bellman equation characterizing dynamic incentive compatibility.
2. If the game has observable actions, establish the existence of the optimal penal codes.
3. Derive the appropriate Hamilton-Jacobi-Bellman equation characterizing the boundary of the set of payoffs attainable in self-enforcing agreements.

These three steps correspond to the three main results of the paper.

First, I characterize when a public agreement is self-enforcing. An agreement is self-enforcing if and only if it satisfies two separate conditions: the One-Stage Deviation in Hidden Actions and the One-Stage Deviation in Observable Actions. The One-Stage Deviation in Hidden Actions is familiar from the literature. In fact, it is exactly the incentive compatibility condition from Sannikov (2007) and it does not contain any money transfers. The One-Stage Deviation in Observable Actions is also quite familiar. It requires that essentially never, either of the players will find it instantaneously profitable to publicly deviate in money transfers alone.

Second, I establish the existence of optimal penal codes in my setting. The notion of optimal penal code was introduced by Abreu (1988). There, an optimal penal code is a tuple of p-SPNEs of a repeated game that deliver to each player his worst possible p-SPNE payoff. That is, an optimal penal code implements the harshest possible subgame-perfect punishments for each of the players. For my second result, I assume that minmaxing each of the players can be *locally* enforced by shifting promised continuation values. (Recall the notion of enforceability of an action profile from Fudenberg et al. (1994) and Sannikov (2007).) I show that if for each player, his stage-game minmaxing profile is enforceable (and under some additional technical restrictions), then there exist a couple of self-enforcing public agreements that *globally* deliver the stage-game minmax payoffs to each of the players. As any self-enforcing agreement must deliver to the players at least their stage-game minmax payoffs, these two agreements indeed implement the harshest punishments.

Third, I characterize the set of payoffs attainable in self-enforcing public agreements. A pair of payoffs w is called *individually rational* if it lies above the players' pure-strategy minmax payoffs from the stage game. A subset S of the set of individually rational payoffs is called *comprehensive* if, for any point $w \in S$, S also contains all individually rational payoffs that may be obtained from w by subtracting a positive linear combination of the money-transfer vectors $(1, -k)$ and $(-k, 1)$. For a subset of individually rational payoffs, prefix ∂_+ denotes the part of the boundary which lies strictly above the minmax lines of the players. Finally, $\mathcal{N}$ denotes the convex hull of pure-strategy Nash equilibria (p-NEs) payoffs of the stage game. My third result states that for any $k \in [0, 1)$, and for any fixed interest rate $r > 0$, whenever an optimal penal code exists, the set K of the payoffs attainable in self-enforcing agreements is precisely the largest convex bounded subset of the set of individually rational payoffs such that (1) K is comprehensive; (2) the boundary of K satisfies the optimality equation of Sannikov (2007) at any point $w \in \partial_+ K \backslash \mathcal{N}$; and (3) $\partial_+ K$ enters the minmax line of Player i, $i = 1, 2$, either at a p-NE payoff or tangent to the corresponding money-transfer vector, namely, $(1, -k)$ for Player 1 and $(-k, 1)$ for Player 2.

The rest of the paper is organized as follows. In section 1.1, I briefly discuss my contributions to the existing literature. In section 2, I introduce the model and provide main definitions. In section 3, I present the main results. In section 4, I describe the dynamics in optimal self-enforcing agreements and consider the cases of fixed-cost and perfect transfers.

1.1 Related Literature

This paper contributes to the existing literature in at least three ways.

First, it adds to the body of work on subgame perfect equilibria of infinitely repeated games. On the one hand, Abreu et al. (1986) propose an algorithm for computing SPNE payoffs of repeated games with imperfectly observable actions in discrete time. Fudenberg et al. (1994) further the understanding of the provision of incentives in such games. Finally, Sannikov (2007) advances the study of the equilibria of such games (at least in the two-player case) by setting them up in continuous time and characterizing in closed form the set of their equilibria payoffs, the optimal provision of incentive, and the dynamics in optimal equilibria. On the other hand, Abreu (1988) develops a method for studying infinitely repeated games with observable actions in discrete time. The current paper makes an effort to combine the techniques of Sannikov (2007) and Abreu (1988) in order to formulate and solve for equilibria of repeated games in continuous time with both perfectly and imperfectly observable actions.

Second, the paper adds to the literature on continuous-time games with observable actions.

Simon and Stinchcombe (1989) provide a discussion of potential technical issues in modeling such games. To resolve these issues, they propose to look at continuous-time strategies as limits of discrete-time strategies for increasingly finer grids. Doing so effectively makes working in continuous time no more attractive than in discrete. Moreover, they impose that players can change their actions only finitely many times. Thus, their method can not apply to the repeated games studied in this paper. Bergin and MacLeod (1993) formulate players' strategies directly in continuous time. They impose a certain inertia restriction on these strategies, which is much less restrictive than the assumptions of Simon and Stinchcombe (1989). However, in Bergin and MacLeod (1993), strategies are indistinguishable if they are the same at almost all times. In particular, their model treats a strategy that prescribes no deviations at all as the same as a strategy that prescribes one observable deviation. Thus, it is questionable whether their model properly accounts for the problem of information in extensive form. In comparison with the current paper, Bergin and MacLeod (1993)'s inertia requires that, stated intuitively, once a player chooses an action, he is stuck with it for a short time. In this paper, the inertia requires that once a player observably deviates from the currently effective outcome, he becomes stuck with his deviating action for a short time. On the one hand, my inertia is more restrictive: I require that the amount of time for which deviators are stuck with their actions is fixed for the whole agreement, whereas Bergin and MacLeod (1993) allow this time to be different after different histories. On the other hand, my inertia is less harsh in that it applies only to deviations from the agreement's outcomes. In particular, in my model the initial path of play is not restricted by the inertia. This grants my model substantial tractability.

In recent years, several papers have tried to incorporate observable actions into continuous-time models. The closest to the current paper is Jiang and Zhang (2019), in which they consider a version of the model that I use with a specific stage game and signal structure. The optimal path of play they suggest coincides with the optimal path found in the current paper. The main difference, however, is that Jiang and Zhang (2019) seem to analyze the model in reduced form without addressing the issues of modeling continuous-time games with observable actions. Hackbarth and Taub (2019) consider a version of the model in Sannikov (2007) in which the players can mutually agree on an exogenous exit option, with this decision being observable. However, their paper does not seem to address the problem of observable actions either. There is also a line of work on durable good monopoly and bargaining in continuous time; see Ortner (2017), Ortner (2019), Chavez (2019), Daly and Green (2018). These authors do acknowledge the issues in modeling observable actions in continuous time. Instead of setting up their games in continuous time completely, they look for stationary outcomes that satisfy certain properties of equilibria of the corresponding discrete-time models.

Finally, by covering the intermediate case, this paper bridges the gap between (i) research on repeated games without transfers, and (ii) research on repeated games with perfect transfers (Fong and Surti (2009), Goldlücke and Kranz (2012), Goldlücke and Kranz (2013)) and relational contracts (Baker et al. (2002), Levin (2003), Rayo (2007)).

2 Model

In this section, I introduce and discuss the main ingredients of the model.

2.1 Basic Setup

The model builds upon the model of continuous-time two-player repeated games with imperfect public monitoring studied in Sannikov (2007).

Two players repeatedly interact in continuous time. At each time $t \in [0, \infty)$, Player i takes a productive action A_t^i from a finite set $\mathcal{A}^i$. These productive actions $A_t = (A_t^1, A_t^2)$ are imperfectly observable by their effect on the evolution of a d-dimensional public-signal process X_t,

$$X_t = \int_0^t \mu(A_s)\, ds + Z_t,$$

where Z_t is a d-dimensional Brownian motion and $\mu : \mathcal{A}^1 \times \mathcal{A}^2 \to \mathbb{R}^d$ is a drift function. The arrival of public information is captured by an exogenously given filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$.

The new feature in my model is that besides the possibility of taking imperfectly-observable productive actions, the players possess an exogenously given technology that allows them to publicly transfer money between each other. Specifically, there is an exogenously given retention parameter $k \in [0, 1)$ characterizing how efficient these transfers are. If at time t, Player i sends the opponent amount $d\Gamma_t^i > 0$, then the opponent receives only $k \cdot d\Gamma_t^i$, with the remaining $(1 - k) \cdot d\Gamma_t^i$ being permanently lost. Denote through Γ_t^i the cumulative process of transfers sent by Player i until time t inclusive.

Suppose that during the play of this interaction, the players take a profile of unobservable actions $(A_t^1, A_t^2)_{\{t \geqslant 0\}}$ and a profile of cumulative public transfers $(\Gamma_t^1, \Gamma_t^2)_{\{t \geqslant 0\}}$. (In what follows, I will always restrict attention to such profiles that $(A_t^1, A_t^2)_{\{t \geqslant 0\}}$ are progressively measurable and $(\Gamma_t^1, \Gamma_t^2)_{\{t \geqslant 0\}}$ are weakly-increasing nonnegative RCLL-processes adapted to $\{\mathcal{F}_t\}_{t \geqslant 0}$.) Player i's random total discounted payoff under the play of this profile is

$$r \int_0^\infty e^{-rt} \big(c_i(A_t^i) dt + b_i(A_t^i) dX_t - d\Gamma_t^i + k d\Gamma_t^{-i} \big) - r\Gamma_0^i + rk\Gamma_0^{-i},$$

for some functions $c_i : \mathcal{A}^i \to \mathbb{R}$ and $b_i : \mathcal{A}^i \to \mathbb{R}^d$, where $r > 0$ denotes the common discount rate of the players.

Denote

$$g_i(A_t) = c_i(A_t^i) + b_i(A_t^i) \mu(A_t).$$

Player i's continuation payoff expected at time t given continuation profile $(A, \Gamma)_{\{s \geqslant t\}} = (A_s^1, A_s^2, \Gamma_s^1, \Gamma_s^2)_{\{s \geqslant t\}}$ then can be written as

$$W_t^i(A,\Gamma) = E_t\Big[r\int_t^\infty e^{-r(s-t)}\big(g_i(A_s) - d\Gamma_s^i + kd\Gamma_s^{-i}\big) - r\Delta\Gamma_t^i + rk\Delta\Gamma_t^{-i}|A_s, s \geqslant t\Big],$$

where $\Delta\Gamma_t = \Gamma_t - \Gamma_{t-}$ if $t > 0$ and $\Delta\Gamma_0 = \Gamma_0$.

2.2 Outcomes

In this subsection, I define public outcomes, from which I eventually will build public agreements.

Within an agreement, an outcome Q describes the recommended continuation path of play starting immediately after the last observed deviation from the previously effective outcome. In particular, Q contains a filtered probability space $(\Omega^Q, \mathcal{F}^Q, \{\mathcal{F}_t^Q\}_{t\geqslant 0}, \mathbf{P}^Q)$ capturing the arrival of public information after the last observed deviation. This information includes the evolution of a d-dimensional public signal X_t^Q and, possibly, the realizations of independent public randomizations. Further, Q specifies a profile $\left(A^{1,Q}, A^{2,Q}\right)$ of recommended hidden actions progressively measurable with respect to $\{\mathcal{F}_t^Q\}_{t\geqslant 0}$, and recommended cumulative money-transfer processes $(\Gamma^{1,Q}, \Gamma^{2,Q})$, which are weakly-increasing nonnegative RCLL-processes adapted to $\{\mathcal{F}_t^Q\}_{t\geqslant 0}$. The measure $\mathbf{P}^Q$ agrees with the profile of recommended hidden actions in such a way that the process $X_t^Q - \int_0^t \mu(A_s^Q)\, ds$ is a standard d-dimensional Brownian motion under $\mathbf{P}^Q$.

More formally, the public information for outcome Q is constructed in the following way:

Definition (Public Information)**.** *For an outcome Q, the public information $\mathcal{P}^Q$ is a filtered probability space $(\Omega^Q, \mathcal{F}^Q, \{\mathcal{F}_t^Q\}_{t\geqslant 0}, \boldsymbol{P}^Q)$, which is constructed as follows:*

1. *Take a filtered probability space $\mathcal{P}^0 = (\Omega^0, \mathcal{F}^0, \{\mathcal{F}_t^0\}_{t\geqslant 0}, \boldsymbol{P}^0)$ to be used for public randomization (take this space rich enough so that $\mathcal{F}^0$ includes the realization of a random variable distributed $U[0,1]$).*

2. *Take a standard d-dimensional Brownian motion X_t on a filtered probability space $\mathcal{P}^X$.*

3. *Take the direct product $\mathcal{P} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \boldsymbol{P})$ of the above filtered probability spaces:*

$$\mathcal{P} = \mathcal{P}^0 \otimes \mathcal{P}^X.$$

4. *Set $\Omega^Q = \Omega$.*

5. *Take a profile $\left(A^{1,Q}, A^{2,Q}\right)$ of recommended hidden actions (which can be any progressively measurable process of hidden actions on $\mathcal{P}$).*

6. *Using Girsanov's theorem, construct the measure $\boldsymbol{P}^Q$ on $(\Omega^Q, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0})$ so that $X_t^Q - \int_0^t \mu(A_s^Q)\, ds$ is a d-dimensional Brownian motion under $\boldsymbol{P}^Q$.*

7. *Finally, define $(\mathcal{F}^Q, \{\mathcal{F}_t^Q\}_{t\geqslant 0})$ as the right-continuous augmentation of $(\mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0})$ under $\boldsymbol{P}^Q$.*

I will say that public information $(\Omega^Q, \mathcal{F}^Q, \{\mathcal{F}_t^Q\}_{t\geqslant 0}, \mathbf{P}^Q)$ *agrees* with profile $\left(A^{1,Q}, A^{2,Q}\right)$ of hidden actions if this information is constructed using $\left(A^{1,Q}, A^{2,Q}\right)$.[2]

Besides recommended hidden actions, outcome Q also specifies recommended money-transfer processes $(\Gamma^{1,Q}, \Gamma^{2,Q})$. I restrict $(\Gamma^{1,Q}, \Gamma^{2,Q})$ to be weakly-increasing nonnegative adapted RCCL-processes on $(\Omega^Q, \mathcal{F}^Q, \{\mathcal{F}_t^Q\}_{t\geqslant 0}, \mathbf{P}^Q)$. Further, I require that processes $(\Gamma^{1,Q}, \Gamma^{2,Q})$ be M*-nonmanipulable* for some $M > 0$ as defined below:

Definition (Well Bounded Process)**.** *Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \boldsymbol{P})$, a weakly-increasing nonnegative adapted RCLL-process Γ_t is said to be well bounded by $M > 0$ if for any finite $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time T,*

$$E^{\boldsymbol{P}}\Big[\int_T^\infty e^{-r(s-T)} d\Gamma_s + \Delta\Gamma_T \Big| \mathcal{F}_T\Big] \leqslant M \quad (\mathcal{F}_T, \boldsymbol{P})\text{-}a.s.$$

Definition (M-Nonmanipulable Processes)**.** *Given public information $(\Omega^Q, \mathcal{F}^Q, \{\mathcal{F}_t^Q\}_{t\geqslant 0}, \boldsymbol{P}^Q)$ that agrees with profile $\left(A^{1,Q}, A^{2,Q}\right)$ of recommended hidden actions, a weakly-increasing nonnegative adapted RCLL-process Γ_t is said to be M-nonmanipulable for some $M > 0$ if for each Player i, $i = 1, 2$, and for any progressively measurable process $\tilde{A}^i$ of hidden actions for Player i, the process Γ_t is well bounded by M under the measure $\boldsymbol{P}(\tilde{A}^i, A^{-i,Q})$ which is obtained from $\boldsymbol{P}^Q$ by changing $A^{i,Q}$ to $\tilde{A}^i$.*

M-nonmanipulability of Γ^{-i}, the money-transfer process for Player $-i$ recommended by outcome Q, guarantees that Player i would not be able to "jam" the public signal by changing his hidden actions so that to make the opponent transfer him in expectation infinite amount of money.

I am now ready to introduce the formal definition of public outcome, which will be the main building block in the construction of public agreements in the next subsection.

Definition (Outcome)**.** *A public outcome $Q = \{\mathcal{P}^Q, A^Q, \Gamma^Q\}$ is public information $\mathcal{P}^Q$ together with recommended processes of hidden actions $\left(A^{1,Q}, A^{2,Q}\right)$ and cumulative money transfers $(\Gamma^{1,Q}, \Gamma^{2,Q})$ such that*

1. *$\left(A^{1,Q}, A^{2,Q}\right)$ are progressively measurable and agree with $\mathcal{P}^Q$;*
2. *$(\Gamma^{1,Q}, \Gamma^{2,Q})$ are weakly-increasing nonnegative adapted RCLL-processes M-nonnmanipulable for some $M > 0$;*

Note that whenever a certain outcome becomes effective during the play, the clock is completely restarted: the time is set to $t = 0$ and the public information begins anew.

[2]Recommended hidden actions are included into the construction of public information for a purely technical reason. With infinite horizon, the hidden-action processes affect which events become measure zero. Thus, the augmentation in the last step of the construction depends of the recommended hidden actions. The augmentation is needed, for example, to apply the Martingale Representation Theorem in the proof of Proposition 1.

2.3 Public Agreements

Having introduced the concept of an outcome in the previous subsection, I am ready to define public agreements, one of the main concepts in the paper. A public agreement is a collection of public outcomes. An agreement proposes to start with some initial outcome Q^0. It also specifies punishment outcomes suggesting the continuation play after any *finite* sequence of observed deviations. Below I introduce an important inertia restriction. Intuitively, it is the restriction on how frequently the players are allowed to *publicly deviate* from the outcomes in an agreement. The inertia guarantees that after essentially any finite history during the play of an agreement, there will be only finitely many observed deviations. In particular, this means that agreements will be well defined: an agreement will be recommending a well-defined continuation play after any finite history possible under the play of that agreement.

Within each outcome of an agreement, I restrict that the players are only allowed to publicly deviate at times when they are prescribed to send the opponents positive transfers, at *permissible times of public deviations.*

Definition (Permissible time of the public Deviation)**.** *Given an outcome $Q = \{\mathcal{P}^Q, A^Q, \Gamma^Q\}$, an $\{\mathcal{F}^Q_t\}_{t\geqslant 0}$-stopping time T is a permissible time of the public deviation for Player i if Player i is supposed to send positive amount of money at T. That is $T < \infty$ implies that $\Gamma^{i,Q}_T$ is right-increasing at T or that $\Gamma^{i,Q}_0 > 0$ and $T = 0$.*

An agreement contains an initial outcome and outcomes specifying punishments after observed public deviations. Fix small $\epsilon > 0$, the parameter of the inertia. The following is the restriction on punishment outcomes that can be employed in an agreement with inertia parameter ϵ:

Inertia Restriction. *If $Q = \{\mathcal{P}^Q, A^Q, \Gamma^Q\}$ is a punishment outcome of an agreement with inertia parameter $\epsilon > 0$, then Q must specify that at the beginning, no player sends positive transfers at least until the first time the public signal moves by ϵ or until ϵ amount of time elapses,*

$$\Gamma^Q_{\tau-} = (0,0), \text{ where } \tau = \min\{t : |X^Q_t| = \epsilon\} \wedge \epsilon.$$

The inertia together with when public deviations are permitted amounts to the following three restrictions on agreements:

1. any deviations in money transfers are ignored if the deviating player is supposed to send zero;

2. the only deviations in money transfers that are considered are to send zero;

3. if a player observably deviates, he is stuck with his deviation for a positive amount of time.

Restrictions 1 and 2 can be shown to be without loss of generality (similar to Abreu (1988)). I introduce them only as a simplification. Restriction 3 is the main restriction. This restriction is automatically satisfied in any discrete-time model. In this continuous-time setting, I impose it directly. Also, the inertia is made sensitive to large moves of the public signal. For a Brownian signal, such moves can happen very quickly with small probability. The sensitivity ensures that the boundary of the payoff set for self-enforcing agreements is easy to characterize and that it does not depend on sufficiently small inertia parameters. For a more detailed discussion of the inertia restriction please refer to Panov (2019).

I am now ready to provide the formal construction of public agreements.

Definition (Public Agreement). *A public agreement $\mathcal{E}$ with inertia parameter $\epsilon > 0$ is a collection of public outcomes which is constructed in the following steps:*

1. *$\mathcal{E}$ specifies the initial outcome Q^0;*

2. *given Q^0, $\mathcal{E}$ specifies all punishment outcomes of level-1, the punishment outcomes after the first observed deviation by Player 1 or Player 2 for all permissible times of public deviations for these players in Q^0;*

3. *for each punishment outcome Q^1 of level-1, $\mathcal{E}$ specifies all punishment outcomes of level-2 following Q^1, the punishment outcomes after the second observed deviation by Player 1 or Player 2 for all permissible times of public deviations for these players in Q^1;*

4. *for each punishment outcome Q^2 of level-2, $\mathcal{E}$ specifies all punishment outcomes of level-3 following Q^2, the punishment outcomes after the third observed deviation by Player 1 or Player 2 for all permissible times of public deviations for these players in Q^2;*

5. *and so on...*

Additionally, there must exist a uniform bound $M > 0$ such that for all outcomes in $\mathcal{E}$, the recommended money-transfer processes are M-nonmanipulable. Pieces of public information from different outcomes in $\mathcal{E}$ are treated as independent of each other.

Pure public strategies for the players are defined only against a given public agreement.

Definition (Pure Public Strategy). *Given a public agreement $\mathcal{E}$ with inertia parameter $\epsilon > 0$, a pure public strategy σ for Player i is a collection of separate rules σ^Q prescribing the behavior in each outcome Q from $\mathcal{E}$. Each σ^Q consist of*

1. *$A^{i,Q,\sigma}$, a process of hidden actions for Player i progressively measurable given the public filtration $\{\mathcal{F}_t^Q\}_{t\geqslant 0}$;*

2. *An $\{\mathcal{F}_t^Q\}_{t\geqslant 0}$-stopping time $T^{i,Q,\sigma}$ prescribing the moment at which Player i announces his public deviation from Q. The stopping time $T^{i,Q,\sigma}$ is restricted to be a permissible time of public deviation for Player i.*

$S^i(\mathcal{E})$ denotes the set of all pure public strategies for Player i against agreement $\mathcal{E}$.

During the play of an agreement, there is always exactly one currently effective outcome recommending the continuation play to the players. The outcome remains effective until the first time T at which either player (possibly both) publicly deviates. A public deviation at time $T(\omega)$ causes the instantaneous hold on money transfers, i.e., sets $\Delta\Gamma^1_T(\omega) = \Delta\Gamma^2_T(\omega) = 0$. Also, the deviation makes the continuation play switch to the new effective outcome, the corresponding punishment outcome prescribed in the agreement.

Suppose that given an agreement $\mathcal{E}$ with inertia parameter $\epsilon > 0$, the players decide to play a profile of pure public strategies (σ^1, σ^2). Because of the inertia restriction and the fact that public deviations are only permissible at times when the deviating player is supposed to send positive amount of money, for any finite time $t > 0$, "with probability 1," there will be only finitely many public deviations observed by time t. Indeed, if there is a finite history such that the players have deviated infinitely many times until time t, then infinitely many times along this history, the public deviations became possible by an ϵ-jump of then effective public signal X^Q. But there exist $c > 0$ such that for any outcome Q and any hidden action profile of the players, ϵ-jump of X^Q before time ϵ happens with probability less than $1 - c$. As public signals across different outcomes are treated as independent, the probability that infinitely many such jumps happened before time t then is at most $(1 - c)^\infty = 0$. Therefore, $\mathcal{E}$ correctly determines the proposed continuation play for essentially any finite public history arising from the play of any pure public strategy profile.

2.4 Promised Continuation Values

I now specify the continuation values promised under the play of a public agreement. As usual, these continuation values are computed assuming that nobody further deviates from the currently proposed path of play.

Suppose the players are playing against an agreement $\mathcal{E}$ and after some history, an outcome Q (either initial or punishment) is effective. Within Q, one can define the process of promised continuation values as the discounted sum of future stage-game payoffs and net money transfers evaluated at time $t \geqslant 0$, similarly to how it is done in Sannikov (2007). Specifically, at time t after the start of Q, Player i's promised continuation value is

$$W_t^{i,Q} = E_t^{\mathbf{P}^Q}\Big[r\int_t^\infty e^{-r(s-t)}\big(g_i(A_s^Q)ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Delta\Gamma_t^{i,Q} + rk\Delta\Gamma_t^{-i,Q}\Big|\mathcal{F}_t^Q\Big].$$

The boundedness of the stage-game payoffs and the well boundedness of the money-transfer processes ensures that one can always find a bounded modification for $W_t^{i,Q}$. Note that $W_t^{i,Q}$ is a random variable. I do not attach any game-theoretic meaning to it. I will only use $W_t^{i,Q}$ throughout derivations. The only continuation value to which I actually attach a game-theoretic meaning and do interpret it as the value from the outcome as assessed by the player is $W^{i,Q}$, the unconditional expectation of $W_0^{i,Q}$ computed at the very beginning of Q:

$$W^{i,Q} := E^{\mathbf{P}^Q}\Big[W_0^{i,Q}\Big].$$

Given an agreement $\mathcal{E}$ with the initial outcome Q^0, define *the expected payoff* $W^{i,\mathcal{E}}$ *promised by* $\mathcal{E}$ to Player i as

$$W^{i,\mathcal{E}} := W^{i,Q_0}.$$

The following is a straightforward adaptation of Proposition 1 from Sannikov (2007) to the current setting:

Proposition 1. *(Representation and Promise Keeping) A bounded stochastic process W_t^i is the process of promised continuation values $W_t^{i,Q}$ of Player i in outcome Q if and only if there exist processes $\beta^{i,Q} = (\beta^{i1,Q}, ..., \beta^{id,Q})$ in $\mathcal{L}^*(\mathcal{P}^Q)$ and a martingale $\tilde{\epsilon}^{i,Q}$ on $\mathcal{P}^Q$ orthogonal to X^Q with $\tilde{\epsilon}_0^{i,Q} = 0$ such that for all $t > 0$, W_t^i satisfies*

$$W_t^i = W_0^i + r\int_0^t \big(W_s^i - g_i(A_s^Q)\big)ds + r\Big(\Gamma_0^{i,Q} + \int_0^t d\Gamma_s^{i,Q}\Big) - r\Big(k\Gamma_0^{-i,Q} + \int_0^t kd\Gamma_s^{-i,Q}\Big) + \\ + r\int_0^t \beta_s^{i,Q}\big(dX_s^Q - \mu(A_s^Q)ds\big) + \tilde{\epsilon}_t^{i,Q}. \quad (1)$$

The proof of Proposition 1 is almost identical to the proof of Proposition 1 from Sannikov (2007) and is left to the reader.

The shorthand form for representation (1) is

$$dW_t^{i,Q} = r\big(W_t^{i,Q} - g_i(A_t^Q)\big)dt + rd\Gamma_t^{i,Q} - rkd\Gamma_t^{-i,Q} + \beta_t^{i,Q}\big(dX_t^Q - \mu(A_t^Q)dt\big) + d\tilde{\epsilon}_t^{i,Q}. \quad (2)$$

Comparing to Sannikov (2007), the new terms in equation (2) are $rd\Gamma_t^{i,Q}$ and $(-rkd\Gamma_t^{-i,Q})$. Intuitively, if at time t, a player sends the opponent G dollars, then his promised continuation value at the very next moment must go up by rG so as to precisely compensate him. At the same time, the opponent's continuation value must go down by rkG to reflect the receipt of the transfer. The new terms capture exactly this intuition.

2.5 The Value of a Strategy

My next task is to define the value of a strategy for a player. Suppose the players are playing against an agreement $\mathcal{E}$. Take Player i and a pure public strategy σ for him. What can be the value of σ evaluated *at the beginning* of some outcome Q from $\mathcal{E}$?

Suppose σ prescribes no public deviations from Q. That is, the stopping time of the deviation

$T^{i,Q}$ is $+\infty$ everywhere. Naturally, one can compute the continuation value of σ at the beginning of Q as the expected discounted sum of payoffs along Q,

$$V(\sigma,Q) := E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^\infty e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big],$$

where $\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})$ is the measure induced in $\mathcal{P}^Q$ by the profile of hidden actions $(A^{i,Q,\sigma},A^{-i,Q})$.

Suppose now that starting from Q, σ prescribes at most one public deviation. Denote by $\tilde{Q}(T,\omega)$ the punishment outcome specified by $\mathcal{E}$ after Player i publicly deviates from Q in state ω at time T. Naturally, one can define the continuation value of σ after this deviation as $V\big(\sigma,\tilde{Q}(T,\omega)\big)$. What about the value of σ evaluated at the beginning of Q? Naively, one might want to write it down as

$$V(\sigma,Q) = E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^{T^{i,Q}} e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big] + \\ + E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[e^{-rT^{i,Q}}\Big(V\big(\sigma,\tilde{Q}(T^{i,Q},\omega)\big) + r\Delta\Gamma_{T^{i,Q}}^{i,Q} - rk\Delta\Gamma_{T^{i,Q}}^{-i,Q}\Big)\Big]. \quad (3)$$

Unfortunately, the second term in the above expression is not well defined generally because $V\big(\sigma,\tilde{Q}(T^{i,Q},\omega)\big)$ is not necessarily a random variable. Thus, the value of such a strategy can not be directly assessed from the point of view of a player, who at stopping time $T^{i,Q}$, only observes the stopped σ-algebra $\mathcal{F}_{T^{i,Q}}^Q$. Because of that, instead of assigning the precise value to $V(\sigma,Q)$, let us assign the upper bound for this value, $V^*(\sigma,Q)$, and the lower bound for this value, $V_*(\sigma,Q)$, by using correspondingly the upper and the lower integrals relative to $\mathcal{F}_{T^{i,Q}}^Q$ for the second term in (3). Formally,

$$V^*(\sigma,Q) := E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^{T^{i,Q}} e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big] + \\ + \Big(E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big)^*\Big[e^{-rT^{i,Q}}\Big(V\big(\sigma,\tilde{Q}(T^{i,Q},\omega)\big) + r\Delta\Gamma_{T^{i,Q}}^{i,Q} - rk\Delta\Gamma_{T^{i,Q}}^{-i,Q}\Big)\Big]$$

and

$$V_*(\sigma,Q) := E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^{T^{i,Q}} e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big] + \\ + \Big(E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big)_*\Big[e^{-rT^{i,Q}}\Big(V\big(\sigma,\tilde{Q}(T^{i,Q},\omega)\big) + r\Delta\Gamma_{T^{i,Q}}^{i,Q} - rk\Delta\Gamma_{T^{i,Q}}^{-i,Q}\Big)\Big],$$

where $\left(E^{\mathbf{P}}\right)^*$ and $\left(E^{\mathbf{P}}\right)_*$ denote the upper and the lower integrals relative to $\mathcal{F}^Q_{T^{i,Q}}$.[3]

Next, for any strategy σ prescribing finitely many observable deviations, define the upper and the lower bounds on its value recursively as

$$V^*(\sigma, Q) := E^{\mathbf{P}(A^{i,Q,\sigma}, A^{-i,Q})}\Big[r \int_0^{T^{i,Q}} e^{-rs}\big(g_i(A^{i,Q,\sigma}_s, A^{-i,Q}_s)ds - d\Gamma^{i,Q}_s + k\, d\Gamma^{-i,Q}_s\big) - r\Gamma^{i,Q}_0 + rk\Gamma^{-i,Q}_0\Big] +$$
$$+ \left(E^{\mathbf{P}(A^{i,Q,\sigma}, A^{-i,Q})}\right)^*\Big[e^{-rT^{i,Q}}\ \Big(V^*\big(\sigma, \tilde{Q}(T^{i,Q}, \omega)\big) + r\Delta\Gamma^{i,Q}_{T^{i,Q}} - rk\Delta\Gamma^{-i,Q}_{T^{i,Q}}\Big)\Big]$$

and

$$V_*(\sigma, Q) := E^{\mathbf{P}(A^{i,Q,\sigma}, A^{-i,Q})}\Big[r \int_0^{T^{i,Q}} e^{-rs}\big(g_i(A^{i,Q,\sigma}_s, A^{-i,Q}_s)ds - d\Gamma^{i,Q}_s + k\, d\Gamma^{-i,Q}_s\big) - r\Gamma^{i,Q}_0 + rk\Gamma^{-i,Q}_0\Big] +$$
$$+ \left(E^{\mathbf{P}(A^{i,Q,\sigma}, A^{-i,Q})}\right)_*\Big[e^{-rT^{i,Q}}\ \Big(V_*\big(\sigma, \tilde{Q}(T^{i,Q}, \omega)\big) + r\Delta\Gamma^{i,Q}_{T^{i,Q}} - rk\Delta\Gamma^{-i,Q}_{T^{i,Q}}\Big)\Big].$$

Finally, for a strategy prescribing arbitrary many observable deviations, define the upper and the lower bounds on its value as

$$V^*(\sigma, Q) := \limsup_{N\to\infty} V^*(\sigma_N, Q),$$

$$V_*(\sigma, Q) := \liminf_{N\to\infty} V_*(\sigma_N, Q),$$

where σ_N is the N-th truncation of σ. That is, σ_N coincides with σ until the N-th public deviation by Player i and follows the actions recommended by the agreement ever after.

The last step is a crucial one and needs to be justified. Indeed, as $N \to \infty$, because of the inertia restriction on how frequently the players can publicly deviate, the strategies σ and σ^N are different either in the event with vanishingly small probability or after the time horizon going to ∞. As the payoffs in the stage game are bounded and all money-transfer processes in $\mathcal{E}$ are uniformly M-nonmanipulable for some $M > 0$, this difference can be effectively ignored in the limit. For the same reason, lim sup and lim inf in the above definitions can be replaced with the usual limits.

[3] For a function $f : \Omega \to \mathbb{R}$ and a σ-algebra $\mathcal{F}$, define the upper and the lower integrals of f relative to $\mathcal{F}$ as $\left(E^{\mathbf{P}}\right)^*(f) := \inf_{\substack{g \text{ is } \mathcal{F}\text{-measurable} \\ \forall\omega\in\Omega, g(\omega)\geqslant f(\omega)}} E^{\mathbf{P}}(g)$ and $\left(E^{\mathbf{P}}\right)_*(f) := \sup_{\substack{g \text{ is } \mathcal{F}\text{-measurable} \\ \forall\omega\in\Omega, g(\omega)\leqslant f(\omega)}} E^{\mathbf{P}}(g)$.

Naturally, $\left(E^{\mathbf{P}}\right)^*(f) \geqslant \left(E^{\mathbf{P}}\right)_*(f)$. Also, $\left(E^{\mathbf{P}}\right)^*(f) = \left(E^{\mathbf{P}}\right)_*(f) \in (-\infty, +\infty)$ if and only if f is $\mathcal{F}$-measurable and integrable, in which case $E^{\mathbf{P}}(f) = \left(E^{\mathbf{P}}\right)^*(f) = \left(E^{\mathbf{P}}\right)_*(f)$.

3 Main Results

In this section, I establish three main results of the paper: the characterization of self-enforcing agreements through a one-stage deviation principle; the existence of optimal penal codes; and the characterization of the set of payoffs attainable in self-enforcing agreements.

3.1 Self-Enforcing Public Agreements

The main concept in my paper is that of *a self-enforcing public agreement*, which is defined as following:

Definition (Self-Enforcing Public Agreement)**.** *A public agreement $\mathcal{E}$ is called self-enforcing if for each of its outcomes $Q \in \mathcal{E}$, no player can find a pure public strategy with the upper bound on the value higher than the promised continuation value when evaluated at the beginning of Q,*

$$\forall Q \in \mathcal{E}, \quad \forall i = 1,2 \quad, \forall \sigma \in S^i(\mathcal{E}), \quad V^*(\sigma, Q) \leqslant W^{i,Q}.$$

The following measurability restriction is a technical restriction on selecting different punishment outcomes in public agreements:

Definition (Measurable Public Agreement)**.** *A public agreement $\mathcal{E}$ is called measurable if for any outcome $Q \in \mathcal{E}$, any Player i, and any permissible time of the public deviation T for Player i, the promised continuation value $W^{i,\tilde{Q}(T)}$ in the resulting punishment is an $\mathcal{F}_T^Q$-random variable.*

Recall representation (1) for promised continuation values given in Proposition 1. The following theorem is the first main result of the paper:

Theorem 1 (One-Stage Deviation Principle)**.** *Let $\mathcal{E}$ be a public agreement. Consider the following restrictions:*

1. *(One-Stage Deviation in Hidden Actions)*

   For each outcome $Q \in \mathcal{E}$, and for any $T \in (0, \infty)$, the inequalities

$$\forall i = 1,2, \quad \forall a_i' \in \mathcal{A}^i, \quad g_i(A_t^Q) + \beta_t^i \mu(A_t^Q) \geqslant g_i(a_i', A_t^{-i,Q}) + \beta_t^i \mu(a_i', A_t^{-i,Q})$$

   are satisfied $(\mathcal{F}_T, \boldsymbol{P}^Q \otimes \lambda[0,T])$-almost surely on $\Omega^Q \times [0,T]$, where $\lambda[0,T]$ is the standard Lebesgue measure on $[0,T]$.

2. *(One-Stage Deviation in Observable Actions)*

   For each outcome $Q \in \mathcal{E}$, for each $i = 1,2$, and for any $\{\mathcal{F}_t^Q\}_{t \geqslant 0}$-stopping time T that is a permissible time of the public deviation to Player i, the instantaneous gain for Player i from disrupting the money transfers and going to the punishment outcome $\tilde{Q}(T,\omega)$ is nonpositive $(\mathcal{F}_T, \boldsymbol{P}^Q)$-almost surely,

$$W_T^{i,Q} \geqslant W^{i,\tilde{Q}(T,\omega)} + r\Delta\Gamma_T^{i,Q} - rk\Delta\Gamma_T^{-i,Q} \qquad (\mathcal{F}_T, \boldsymbol{P}^Q)\text{-}a.s.$$

Then:

- *(Sufficiency) If $\mathcal{E}$ satisfies restrictions* 1 *and* 2*, it is self-enforcing.*
- *(Necessity of* 1*) If $\mathcal{E}$ does not satisfy restriction* 1*, it is not self-enforcing. Moreover, there exists an outcome $Q \in \mathcal{E}$ and a strategy σ for some Player i such that*

$$V^*(\sigma, Q) = V_*(\sigma, Q) > W^{i,Q}.$$

- *(Necessity of* 2*) If $\mathcal{E}$ does not satisfy restriction 2, it is not self-enforcing. Moreover, if $\mathcal{E}$ is measurable, then there exists an outcome $Q \in \mathcal{E}$ and a strategy σ for some Player i such that*

$$V^*(\sigma, Q) = V_*(\sigma, Q) > W^{i,Q}.$$

Proof. See Appendix A.

□

Theorem 1 provides necessary and sufficient conditions for a public agreement to be self-enforcing: a public agreement is self-enforcing if and only if it satisfies the One-Stage Deviation in both hidden and observable actions.

Recall that for an agreement to be self-enforcing, there should be no deviating strategy for either of the players, with the upper bound on the value, rather than the expected value, higher than the value promised by the agreement. This may seem too restrictive. Fortunately, Theorem 1 also establishes that for measurable agreements, this restriction is without loss: if a measurable agreement is not self-enforcing, then there exists a deviating strategy for some player that is a strictly profitable deviation in the sense of the usual expected values. If one wishes, they can restrict attention only to measurable agreements without eliminating any of the supportable outcomes. Indeed, in the next subsection, I consider the optimal penal codes that are pairs of measurable agreements. Any outcome of a self-enforcing agreement can also be supported as an outcome of a self-enforcing agreement with the punishments from an optimal penal code. Thus, any outcome of a self-enforcing agreement can be supported as an outcome of a measurable self-enforcing agreement.

3.2 Optimal Penal Codes

I now turn to the problem of constructing optimal punishments in self-enforcing agreements. Abreu (1988) proves the existence of the optimal penal codes in his discrete-time setting. There, an optimal penal code is a pair of punishment outcomes Q^1 and Q^2, which punish observable deviations by Player 1 and Player 2 correspondingly, such that using them alone, one can construct two p-SPNE's, $\mathcal{E}^1$ and $\mathcal{E}^2$, delivering the worst possible p-SPNE payoffs to Player 1 and to Player 2. In this subsection, I prove an analogous result for self-enforcing public agreements in my continuous-time setting under some additional restrictions.

Denote by $K(\epsilon)$ the set of payoffs attainable in self-enforcing public agreements with inertia parameter $\epsilon > 0$. The next lemma shows that the sets $K(\epsilon)$ are decreasing in ϵ.

Lemma 1 (Monotonicity). *For any $\epsilon_1 > \epsilon_2 > 0$,*

$$K(\epsilon_1) \subseteq K(\epsilon_2).$$

Proof. See Appendix B.1. □

Consider the stage game G in hidden actions played by the players in the current setting. The set of players is $N = \{1, 2\}$, the set of actions for Player i is $\mathcal{A}_i$, the payoff functions are g_i,

$$G = \left\{N, (\mathcal{A}_i)_{i\in N}, (g_i)_{i\in N}\right\}.$$

Denote by $\underline{v}_i$ *the pure-strategy minmax payoff* of Player i in G,

$$\underline{v}_i = \min_{a_{-i}\in\mathcal{A}_{-i}} \max_{a_i\in\mathcal{A}_i} g_i(a_i, a_{-i}).$$

A profile of pure actions that delivers to Player i his mixmax payoff is called *a profile minmaxing* Player i. *The minmax line* for Player i is the straight line in the space of players' payoffs $(w_1, w_2) \in \mathbb{R}^2$ given by the equation $w_i = \underline{v}_i$. The pure-strategy minmax payoff of a player can be interpreted as the player's individual rationality constraint against any pure action of the opponent in the stage game. In my repeated setting, I still can interpret it as the player's individual rationality constraint, the per-period average expected payoff that can be guaranteed by the player against any process of pure hidden actions and money transfers of the opponent. To guarantee this payoff, the player should simply never transfer any money to the opponent and always keep playing the myopic best-response hidden action against the current hidden action of the opponent. The following lemma establishes that any self-enforcing agreement must deliver to both players individually rational payoffs:

Lemma 2 (Individual Rationality). *Any self-enforcing agreement $\mathcal{E}$ delivers to each Player i the expected payoff at least as high as his minmax payoff in the stage game G,*

$$W^{i,\mathcal{E}} \geqslant \underline{v}_i.$$

Moreover, for any outcome $Q \in \mathcal{E}$, and for any stopping time τ, not necessary a permissible time of the public deviation for Player i,

$$W_\tau^{i,Q} \geqslant \underline{v}_i + r\Delta\Gamma_\tau^{i,Q} - rk\Delta\Gamma_\tau^{-i,Q} \qquad \boldsymbol{P}^Q\text{-a.s.}$$

Proof. See Appendix B.2. □

Lemma 2 establishes that the worst payoffs deliverable to the players in self-enforcing agreements must be at least their static minmax payoffs. One can ask whether there exist self-enforcing agreements for each of the players that deliver them precisely this lower bound. I answer this in the affirmative under several additional assumptions. Specifically, in the remainder of the paper, I assume that:

Assumption 1. *There exists* $\epsilon_0 > 0$ *and* $(w_1, w_2) \in K(\epsilon_0)$ *such that* $w_1 > \underline{v}_1$ *and* $w_2 > \underline{v}_2$.

Assumption 1 is guaranteed to be satisfied if there is a p-NE of the stage game with higher than minmax payoffs or if the set of p-PPEs from Sannikov (2007) has nonempty interior.

Recall the definitions of enforceable action profiles and enforceability along hyperplanes from Fudenberg et al. (1994) and Sannikov (2007):

Definition. *A* $2 \times d$ *matrix*

$$B = \begin{bmatrix} \beta^1 \\ \beta^2 \end{bmatrix} = \begin{bmatrix} \beta^{11} & \dots & \beta^{1d} \\ \beta^{21} & \dots & \beta^{2d} \end{bmatrix}$$

enforces action profile $a \in \mathcal{A}$ *if for* $i = 1, 2$,

$$\forall a_i' \in \mathcal{A}^i, \quad g_i(a) + \beta^i \mu(a) \geqslant g_i(a_i', a_{-i}) + \beta^i \mu(a_i', a_{-i}).$$

An action profile $a \in \mathcal{A}$ *is enforceable if there exists some matrix* B *that enforces it.*

Definition. *A vector of volatilities* $\phi \in \mathbb{R}^d$ *enforces action profile* $a \in \mathcal{A}$ *along vector* $\boldsymbol{T} = (t_1, t_2)$ *if the matrix*

$$B = \boldsymbol{T}^\top \phi = \begin{bmatrix} t_1\phi_1 & \dots & t_1\phi_d \\ t_2\phi_1 & \dots & t_2\phi_d \end{bmatrix}$$

enforces a. *Of all vectors* ϕ *that enforce* a *along* $\boldsymbol{T}$, *let* $\phi(a, \boldsymbol{T})$ *be the one with the smallest length.*

Naturally, any p-NE profile a is enforceable with $\phi(a, \mathbf{T}) = (0, 0)$ for any $\mathbf{T}$.

Consider further the following assumptions:

Assumption 2. *All action profiles* $(a_1, a_2) \in \mathcal{A}^1 \times \mathcal{A}^2$ *of the stage game are pairwise identifiable, i.e., the spans of the* $d \times (|\mathcal{A}^1| - 1)$ *matrix* $M_1(a)$ *with columns* $\mu(a_1', a_2) - \mu(a)$, $a_1' \neq a_1$ *and the* $d \times (|\mathcal{A}^2| - 1)$ *matrix* $M_2(a)$ *with columns* $\mu(a_1, a_2') - \mu(a)$, $a_2' \neq a_2$ *intersect only at the origin.*

Assumption 3. *Either*

1. *for all* $i = 1, 2$ *and* $a_i \in \mathcal{A}^i$, *the static best response to* a_i *is unique or*

2. *for all* $a \in \mathcal{A}$, *the spans of* $M_1(a)$ *and* $M_2(a)$ *are orthogonal.*

Assumption 4. *For each player, at least one of the profiles minmaxing him is enforceable.*

Assumptions 2 and 3 are precisely the assumptions used in Sannikov (2007). In particular, these assumptions guarantee that an enforceable action profile is enforceable along all regular vectors. Moreover, an enforceable action profile a is enforceable along vector $\mathbf{T}$ with $t_i = 0$ if and only if a_i is a best response to a_{-i} in the stage game G. Assumption 4 is a new and the most crucial one. It requires that at least locally, one can provide incentives via shift in promised continuation values to each of the players to minmax his opponent. Still this restriction is much weaker than requiring that minmaxing can be incentivized forever.

Consider any two punishment outcomes Q^1 and Q^2 satisfying the inertia restriction for some parameter $\epsilon > 0$. Define the two agreements $\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$, constructed from Q^1 and Q^2, as follows:

- $\mathcal{E}^1(Q^1, Q^2)$ proposes Q^1 as the initial outcome and then at any time of a public deviation, proposes to start the punishment outcome Q^i if the deviation was made by Player i;
- $\mathcal{E}^2(Q^1, Q^2)$ proposes Q^2 as the initial outcome and then at any time of a public deviation, proposes to start the punishment outcome Q^i if the deviation was made by Player i;
- in both $\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$, if public deviations are made by both players simultaneously, the prescribed punishment is Q^1.

Notice that for any two punishment outcomes Q^1 and Q^2, the agreements $\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$ are measurable.

The following theorem establishes the existence of the optimal penal codes in the current setting. It is my second main result.

Theorem 2 (Optimal Penal Code). *Suppose that Assumptions 1, 2, 3, and 4 are satisfied. Then there exist $\bar{\epsilon} > 0$ and public outcomes Q^1 and Q^2 such that for any $\epsilon \in (0, \bar{\epsilon})$,*

1. *Q^1 and Q^2 are punishment outcomes with inertia parameter ϵ;*
2. *$\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$ are self-enforcing public agreements;*
3. *$\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$ deliver the minmax payoffs to Players 1 and 2 correspondingly,*

$$\forall i = 1, 2, \quad W^{i, \mathcal{E}^i(Q^1, Q^2)} = \underline{v}_i.$$

Proof. See Appendix B.4 for a constructive proof. □

3.3 Characterization of the Payoff Set

In this subsection, I provide the characterization of $K(\epsilon)$, the set of payoffs attainable in self-enforcing public agreements for sufficiently small inertia parameters ϵ.

Denote by IR the set of all individually rational payoffs, $IR = \{(w_1, w_2) \in \mathbb{R}^2 : |(w_1 \geqslant \underline{v}_1) \wedge (w_2 \geqslant \underline{v}_2)\}$. In the previous subsection, I showed that $K(\epsilon)$ consists of individually rational payoffs,

$K(\epsilon) \subseteq IR$. For any pair of payoffs $w = (w_1, w_2) \in \mathbb{R}^2$, define the set $C(w)$ as the set of all payoffs that can be obtained from w by subtracting positive linear combinations of the money-transfer vectors $(1, -k)$ and $(-k, 1)$,

$$C(w) = \left\{(x_1, x_2) \in \mathbb{R}^2 | \ \exists \lambda_1 \geqslant 0 \ \exists \lambda_2 \geqslant 0 : \ (x_1, x_2) = (w_1, w_2) - \lambda_1(1, -k) - \lambda_2(-k, 1)\right\}.$$

The notion of *a comprehensive set* then is defined as follows:

Definition (Comprehensive Set). *A subset S of the set of individually rational payoffs IR is called comprehensive if*

$$\forall w \in S, \ C(w) \cap IR \subseteq S.$$

The next lemma shows that $K(\epsilon)$ is a comprehensive subset of IR.

Lemma 3 (Comprehension). *Under Assumptions 1, 2, 3, and 4, there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon})$, the set $K(\epsilon)$ is comprehensive.*

Proof. Take the initial outcome of a self-enforcing agreement $\mathcal{E}$ with payoffs $(w_1, w_2) \in K(\epsilon)$. For individually rational payoffs $(w_1', w_2') = (w_1, w_2) - \alpha_1(1, -k) - \alpha_2(-k, 1)$, construct the outcome, which at the beginning, requires Player 1 to send α_1 amount of money and Player 2 to send α_2 amount of money, and then implements the initial outcome of $\mathcal{E}$. Support this outcome by an optimal penal code from Theorem 2. By Theorem 1, this agreement is also self-enforcing with uniformly non-manipulable outcomes. □

Lemma 4 (Stabilization). *Under Assumptions 1, 2, 3, and 4, there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$, the set $K(\epsilon)$ is the same.*

Proof. By Theorem 1, the set of outcomes supportable in self-enforcing agreements is the same for all inertia parameters, for which there exists an optimal penal code. The rest follows from Theorem 2. □

Lemma 5 (Convexity). *For any $\epsilon > 0$, the set $K(\epsilon)$ is convex.*

Proof. By the standard argument of convexification through an initial public randomization. □

Lemma 6 (Inclusion). *For any $\epsilon > 0$, the set $K(\epsilon)$ includes p-PPE payoffs from Sannikov (2007).*

Proof. Take any p-PPE from Sannikov (2007) resulting in an outcome Q. Construct the agreement $\mathcal{E}$ which specifies Q as the initial outcome. As Q prescribes no positive transfers, there are no public deviations allowed for the players. Hence, $\mathcal{E}$ does not need to specify punishment outcomes. Conditions of Theorem 1 then simplify to the incentive compatibility conditions of Proposition 2 in Sannikov (2007), which are supposed to be satisfied for Q. Also, as there are no punishment outcomes, the inertia restriction is vacuous. Q.E.D. □

Denote by $\partial_+ K(\epsilon)$ the part of the boundary of $K(\epsilon)$ which lies strictly above the minmax lines of the players. Take any point $w = (w_1, w_2) \in \partial_+ K(\epsilon)$. Let $\mathbf{T}(w)$ and $\mathbf{N}(w)$ denote the unit tangent and outward normal vectors for $\partial_+ K(\epsilon)$ at w. As $K(\epsilon)$ is convex, these vectors are uniquely defined for all but at most countably many points of $\partial_+ K(\epsilon)$. Let $\kappa(w)$ be the curvature of $\partial_+ K(\epsilon)$ at w. Recall that $\phi(a, \mathbf{T})$ denotes the vector of volatilities that enforces action profile a along vector T and has the smallest length. If a is not enforceable along $\mathbf{T}$, set $|\phi(a, \mathbf{T})| = \infty$. Also, let $\mathcal{A}^N$ be the set of pure-strategy Nash equilibria (p-NEs) of the stage game G. Let $\mathcal{N}$ be the convex hull of the payoffs from $\mathcal{A}^N$. The following equation is the *optimality equation* of Sannikov (2007):

$$\kappa(w) = \max \Big\{ 0; \max_{a \in (\mathcal{A}^1 \times \mathcal{A}^2) \setminus \mathcal{A}^N} \frac{2\mathbf{N}(w)(g(a) - w)}{r|\phi(a, \mathbf{T}(w))|^2} \Big\}. \tag{4}$$

Sannikov (2007) shows that in his setting, the boundary of the set of p-PPE payoffs satisfies the optimality equation at each point outside of $\mathcal{N}$. The following is an analogous result for my model:

Lemma 7 (Optimality Equation). *Under Assumptions 1, 2, 3, and 4, for any $\epsilon > 0$, at all points outside of $\mathcal{N}$, $\partial_+ K(\epsilon)$ satisfies the optimality equation (4). Moreover, for each $i = 1, 2$, $\partial_+ K(\epsilon)$ enters the minmax line of Player i either at payoffs corresponding to a p-NE of the stage game or tangent to the corresponding money-transfer vector,* $(1, -k)$ *for Player* 1 *and* $(-k, 1)$ *for Player* 2.

Proof. The proof is similar to the proof of Proposition 5 in Sannikov (2007). In fact, the proof that the curvature of $\partial_+ K(\epsilon)$ can not be smaller than the one prescribed by the optimality equation is almost exactly the same. The proof that the curvature can not be greater than the one in the optimality equation, i.e., "the escape argument", differs in the current setting by the introduction of pushes of continuation values caused by the money transfers. However, as $K(\epsilon)$ is comprehensive, at any point along $\partial_+ K(\epsilon)$, the outward normal vector is positively correlated with the money-transfer pushes. Thus, these pushes can only make the escape argument more compelling. See Appendix C.1 for the formal argument.

□

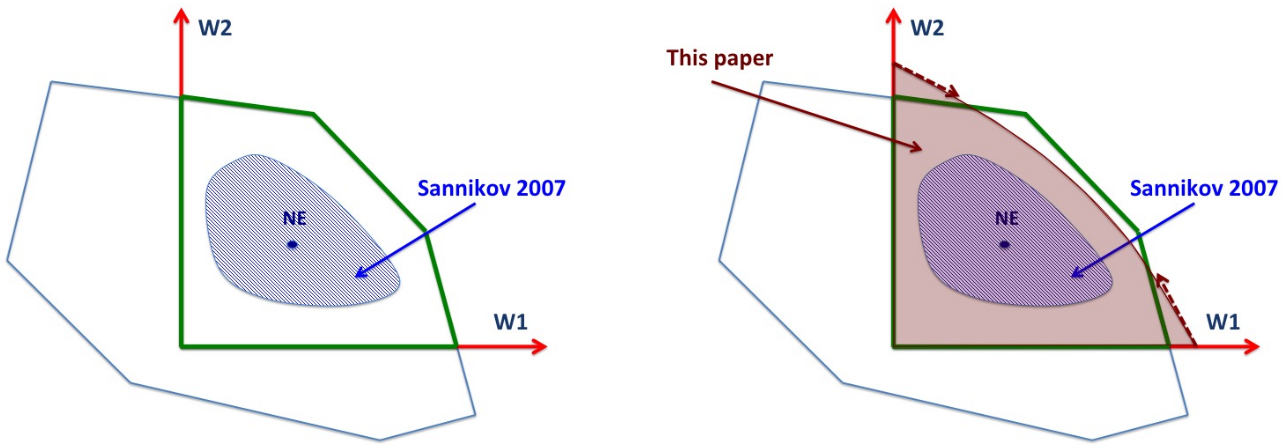
The above lemmata are summarized in the theorem below, which is my third main result.

Theorem 3 (Payoff-Set Characterization). *Under Assumptions 1, 2, 3, and 4, for any $k \in [0, 1)$, there exists $\bar{\epsilon} > 0$ such that for any inertia parameter $\epsilon \in (0, \bar{\epsilon})$, the set $K(\epsilon)$ is the largest compact set that satisfies the following properties:*

1. *$K(\epsilon)$ is a convex and comprehensive subset of the set of individually rational payoffs;*

2. *at all points outside of $\mathcal{N}$, $\partial_+ K(\epsilon)$ satisfies the optimality equation (4), and for each $i = 1, 2$, $\partial_+ K(\epsilon)$ enters the minmax line of Player i either at payoffs corresponding to a p-NE of the stage game or tangent to the corresponding money-transfer vector,* $(1, -k)$ *for Player* 1 *and* $(-k, 1)$ *for Player* 2.

Proof. See Appendix C.2. □



(a) the p-PPE payoff set $\mathcal{S}$ from Sannikov (2007).

(b) $K(\epsilon)$ in the current setting.

Figure 1: Payoffs sets.

Figure 1 compares schematically the set $\mathcal{S}$ of p-PPE payoffs from Sannikov (2007) (Figure 1a) and the corresponding set $K(\epsilon)$ from the current setting (Figure 1b). The blue polygon on both pictures corresponds to the boundary of the convex hull of the stage-game payoffs; the red lines are the players' minmax lines; the green polygon is the boundary of the set $\mathcal{V}^*$, the set of individually rational and feasible-without-transfers payoffs. The blue solid shape in Figure 1a is the set $\mathcal{S}$, the red solid shape in Figure 1b is the corresponding set $K(\epsilon)$. Note that $\mathcal{S}$ does not reach the players' minmax lines unless there are p-NE payoffs on them. Also, $\mathcal{S}$ must lie inside of $\mathcal{V}^*$. In contrast, in the current setting, the set $K(\epsilon)$ reaches both minmax lines as long as the conditions of Theorem 2 are satisfied. Also, $K(\epsilon)$ may extend outside of $\mathcal{V}^*$ as the feasible set is larger when money transfers are available. The positive part of the boundary of $K(\epsilon)$, $\partial_+ K(\epsilon)$, is smooth at all points outside of $\mathcal{N}$ and enters the minmax lines of the players either at p-NE payoffs or parallel to the money-transfer vectors (the red dashed vectors in Figure 1b). Finally, $\partial_+ K(\epsilon)$ typically has strictly positive curvature outside of $\mathcal{N}$. The only exception to that that may be is if $\partial_+ K(\epsilon)$ contains a straight segment, which starts at a player's mixmax line, ends at a p-NE payoff, and is parallel to the money-transfer vector of that player.

4 Discussion

In this section, I discuss the dynamics of in the optimal agreements in my main model, and also consider models with fixed-cost and perfect transfers.

4.1 Optimal Agreements

I now discuss the dynamics in the optimal self-enforcing public agreements. Figure 2 depicts schematically a typical path of continuation values along the initial outcome and punishments

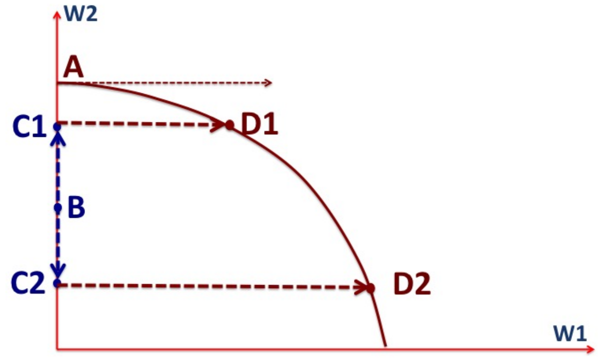


Figure 2: Punishing a deviation from Player 1, $k = 0$.

in an efficient self-enforcing agreement in the case of pure money burning, $k = 0$ (the picture for $0 < k < 1$ looks similarly).

Unless there is a p-NE payoff on $\partial_+ K(\epsilon)$, an agreement that delivers payoffs $w \in \partial_+ K(\epsilon)$ starts with $W_0 = w$ and supports players' incentives by the shift of promised continuation values along $\partial_+ K(\epsilon)$ without costly transfers. The recommended profiles of hidden actions and volatilities of continuation values are determined uniquely by the optimality equation (4). This continues until the promised values hit the minmax line of either player. For example, point A in Figure 2 is the point at which the continuation values hit the minmax line of Player 1. At point A, the agreement introduces the reflective boundary for the process of promised continuation values following the SDE from Proposition 1. To implement this reflective boundary, the agreement asks Player 1 to burn (transfer in case $0 < k < 1$) money so as to match the cumulative amount of money burnt, Γ_t^1, with the compensating process of the reflected continuation values. In particular, money transfers will be happening in infinitesimal installments and only after extreme histories, when it is no longer possible to support incentives by the shift of promised continuation values without violating the individual rationality constraint of Player 1.

Suppose that at point A, Player 1 announces a public deviation. In that case, the agreement will go to the stage of punishing Player 1. This can be done using the construction from Theorem 2. An alternative punishment is shown in Figure 2. The punishment starts by moving the continuation values to point B. This will upset the promises made to Player 2, but this is permissible since Player 2 is not the deviating player. The punishment outcome then supports minmaxing Player 1 by moving the promised continuation values along the minmax line of Player 1 until they hit either C_1 or C_2. At C_1, Player 1 is asked to burn money so as to jump to D_1. Similarly, at C_2, Player 1 is asked to burn money so as to jump to D_2. The punishment outcome then is concatenated with the initial outcomes of the efficient agreements that deliver D_1 or D_2 correspondingly. Theorem 1 ensures that the constructed agreement is self-enforcing.

There are two more things I want to say about the efficient self-enforcing agreements in the current setting. First, an efficient p-PPE in Sannikov (2007) is typically supported by the evolution of promised continuation values that are eventually driven into the area Pareto dominated by other p-PPE payoffs. This may raise concerns regarding the renegotiation proofness of such p-PPEs. In contrast, in the current setting, on the path of play, the promised continuation values of an efficient agreement will always stay on $\partial_+ K(\epsilon)$, the Pareto frontier of $K(\epsilon)$. Thus, the renegotiation-proofness concerns are less severe in my model. Of course, punishing observable deviations still requires the continuation values to plunge into Pareto-dominated areas. However, the "depth" of such plunges may be made arbitrary small by considering the inertia parameters close to zero. Second, the dynamics in the efficient agreements with costly transfers are in sharp contrast with the dynamics in the efficient equilibria in repeated games with perfect transfers (such as Levin (2003), Goldlücke and Kranz (2012)). With perfect transfers, the timing of transfers is not important. Thus, it may be efficient to use them frequently (for example, at the end of every period). With costly transfers, it is optimal to postpone them for as long as possible. Thus, costly transfers are used rarely, only when the individual rationality constraint of either player becomes binding.

4.2 Fixed-Cost Transfers

In the base model, I assume that costs of money transfers are proportional to the amount of money sent. Alternatively, one can consider a version with costs of transfers being fixed. Specifically, suppose the players' transfer technology is characterized by an exogenously given transfer cost $c > 0$. If at time t, Player i wants to deliver $G > 0$ amount of money to the opponent, he has to pay $G + c$. The formal analysis of this model can be done by following essentially the same steps as for the base model, albeit with some minor differences. The first difference is in the existence of the optimal penal codes: the existence can be shown, provided transfer costs are sufficiently small. The second difference is in the definition of a comprehensive set. Precisely, for any pair of payoffs $w = (w_1, w_2) \in \mathbb{R}^2$, define the set

$$\hat{C}(w, c) = \Big\{(x_1, x_2) \in \mathbb{R}^2 \Big| \ (x_1 + x_2 \leqslant w_1 + w_2 - c) \wedge \Big[(x_1 \leqslant w_1 - c) \vee (x_2 \leqslant w_2 - c)\Big]\Big\}.$$

The notion of *a fixed-cost-c comprehensive set* is defined as follows:

Definition (Fixed-Cost-c Comprehensive Set)**.** *A subset S of the set of individually rational payoffs IR is called fixed-cost-c comprehensive if*

$$\forall w \in S, \ \hat{C}(w, c) \cap IR \subseteq S.$$

Denote by $K(\epsilon, c)$ the set of payoffs attainable in self-enforcing agreements with inertia parameter ϵ when fixed costs of money transfers are $c > 0$. The payoff-set characterization for fixed-cost transfers can be formulated as follows:

Theorem 3′ (Payoff-Set Characterization for Fixed-Cost Transfers)**.** *Under Assumptions 1, 2, 3, and 4, there exists $\bar{c} > 0$ such that for any fixed costs of transfers $c \in (0, \bar{c})$, there exists $\bar{\epsilon}(c) > 0$ such* *that for any inertia parameter $\epsilon \in (0, \bar{\epsilon}(c))$, the set $K(\epsilon, c)$ is the largest compact set that satisfies the following properties:*

1. *$K(\epsilon, c)$ is a convex and a fixed-cost-c comprehensive subset of the set of individually rational payoffs;*

2. *at all points outside of $\mathcal{N}$, $\partial_+ K(\epsilon, c)$ satisfies the optimality equation (4), and $\partial_+ K(\epsilon, c)$ enters the minmax line of each player either at payoffs (w_1, w_2) corresponding to a p-NE of the stage game with $w_1 + w_2 \geqslant \max_{(x_1,x_2)\in K(\epsilon,c)} (x_1 + x_2) - c$ or at a point (w_1, w_2) with $w_1 + w_2 = \max_{(x_1,x_2)\in K(\epsilon,c)} (x_1 + x_2) - c$.*

As the proof of Theorem 3′ does not seem to add any considerable insights relative to the proof of Theorem 5, it is omitted. With fixed-cost transfers, the dynamics in the efficient agreements are quite similar to the efficient dynamics with proportional costs. The incentives to cooperate are supported whenever possible by the shifts of promised continuation values. Costly transfers are used only after extreme histories when the promised continuation value of either player hits his individual rationality constraint. The only difference is that with fixed costs, the transfers required at this point are substantial in size so that to move the play immediately to the continuation values with the highest supportable sum of payoffs. This differs from the optimal transfers with proportional costs, which are used in small installments to just reflect from the minmax lines. The reason is that with fixed costs, it is optimal to combine all costly transfers into a single transaction to save on the transaction fee.

4.3 Perfect Transfers

It is reasonable to consider a version of the model with perfect transfers to parallel this continuous setting with discrete-time models of, for example, Levin (2003) and Goldlücke and Kranz (2012).

First, consider the version, in which besides hidden productive actions, the players have access to perfect transfers *only*. Denote by (a_1^*, a_2^*) the most efficient enforceable profile (the one that maximizes the players' total surplus). Let $M^* = g_1(a_1^*, a_2^*) + g_2(a_1^*, a_2^*)$ be the maximal total surplus that is enforceable. Denote by $L(\epsilon)$ the set of payoffs attainable in self-enforcing agreements with inertia parameter ϵ with perfect transfers. The following theorem establishes a lower bound on $L(\epsilon)$ and is parallel to the results from Levin (2003):

Theorem 4 (Cf. Levin (2003))**.** *Let (w_1, w_2) be a p-NE payoff in the stage game G. Then $L(\epsilon)$ contains segment $T(w_1, w_2)$, where*

$$T(w_1, w_2) = \{(x_1, x_2) \in \mathbb{R}^2 | (x_1 + x_2 = M^*) \wedge (x_1 \geqslant w_1) \wedge (x_2 \geqslant w_2)\}.$$

Proof. $T(w_1, w_2)$ can be supported by enforcing the efficient action (a_1^*, a_2^*) along it. The transfers can be requested when the values hit either of the end points $(x_i = w_1)$ for $i = 1, 2$. The transfers can be such that they will send the values to the midpoint of $T(w_1, w_2)$. The transfers are supported by the threat of reverting forever to the static equilibrium with payoffs (w_1, w_2). □

Remarkably, if players can transfer money only perfectly, the set of supportable payoffs may be quite restricted. If besides perfect transfers, some costly transfers are available, then the set of supportable payoffs can become larger. The reason is that while inefficient transfers are never used optimally on the path of play, they can be quite handy in constructing punishment outcomes. For a certain range of payoffs, punishments may be constructed without costly transfers, by simply requiring players to take inefficient actions. However, in general, costly transfers may substantially expand the set of punishment outcomes, and therefore the set of outcomes supportable in self-enforcing agreements.

Because of that, consider now the version in which, like in Goldlücke and Kranz (2012), the players can take hidden actions, perfectly transfer money between each other, and also have access to money burning, $k = 0$. Outcomes in agreements now recommend not only hidden actions A^i and cumulative money-transfer processes Γ^i, but also cumulative money-burning processes M^i. Other than that, this version is similar to the base model. Denote by $GK(\epsilon)$ the set of payoffs that can be supportable in self-enforcing agreements with inertia parameter ϵ in this version. The following theorem provides the characterization of $GK(\epsilon)$ for sufficiently small $\epsilon > 0$:

Theorem 5 (Cf. Goldlücke and Kranz (2012)). *Under Assumptions 1, 2, 3, and 4, there exists $\bar{\epsilon} > 0$ such that for any inertia parameter $\epsilon \in (0, \bar{\epsilon})$, the set $GK(\epsilon)$ is the triangle*

$$GK(\epsilon) = \left\{(x_1, x_2) \in \mathbb{R}^2 | (x_1 + x_2 \leqslant M^*) \wedge (x_1 \geqslant \underline{v}_1) \wedge (x_2 \geqslant \underline{v}_2)\right\}.$$

The proof is a straightforward adaptation of the proof of Theorem 5 and so is omitted. It is important, however, to empathize the major difference between the cases of perfect and costly transfers: Perfect transfers can be used optimally at any time provided the promised payoffs stay above the minmax lines. With costly transfers, there is an additional trade-off between providing incentives through money transfers today and postponing their costs into the future. Optimally, the use of costly transfers is delayed for as long as possible.

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A Proof of Theorem 1

`Sufficiency.`

Take any agreement $\mathcal{E}$. Suppose that it satisfies both restrictions of the One-Stage Deviation Principle. We will prove that $\mathcal{E}$ is self-enforcing. Indeed, consider any strategy σ for any Player i. As the upper bound on the value of σ is the limsup of the upper bounds on the values of its finite truncations, it is sufficient for us to check that $V^*(\sigma, Q) \leqslant W^{i,Q}$ for any $Q \in \mathcal{E}$ and for any σ that prescribes only finitely many observable deviations. We do so by induction in L, the number of observable deviations prescribed by σ.

Base: $L = 0$. Suppose σ does not prescribe any observable deviation. Then $V^*(\sigma, Q) \leqslant W^{i,Q}$ may be proven using the One-Stage Deviation in Hidden Actions restriction alone. In fact, the proof essentially repeats the proof of Proposition 2 from Sannikov (2007) because the money-transfer processes cancel out when we evaluate the effect on Player i's payoff caused by the change in hidden-action profile!

Induction: Suppose that $V^*(\sigma, Q) \leqslant W^{i,Q}$ for any σ prescribing less that $L = l$ observable deviations. Prove for σ that prescribes $L = l$ observable deviations. Take any outcome $Q \in \mathcal{E}$. Recall the definition of $V^*(\sigma, Q)$,

$$V^*(\sigma,Q) = E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^{T^{i,Q}} e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big] + \\ + \Big(E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big)^*\Big[e^{-rT^{i,Q}}\ \Big(V^*\big(\sigma,\tilde{Q}(T^{i,Q},\omega)\big) + r\Delta\Gamma_{T^{i,Q}}^{i,Q} - rk\Delta\Gamma_{T^{i,Q}}^{-i,Q}\Big)\Big].$$

Starting from the punishment $\tilde{Q}\big(T^{i,Q}\big)$ that follows immediately after Player i deviates from Q at $T^{i,Q}$, σ prescribes at most $l-1$ observable deviations. By the induction hypothesis then, $V^*\big(\sigma, \tilde{Q}(T^{i,Q},\omega)\big) \leqslant W^{i,\tilde{Q}(T^{i,Q},\omega)}$. Therefore,

$$V^*(\sigma,Q) \leqslant E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^{T^{i,Q}} e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big] + \\ + \Big(E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big)^*\Big[e^{-rT^{i,Q}}\ \Big(W^{i,\tilde{Q}(T^{i,Q},\omega)} + r\Delta\Gamma_{T^{i,Q}}^{i,Q} - rk\Delta\Gamma_{T^{i,Q}}^{-i,Q}\Big)\Big].$$

Applying the One-Stage Deviation in Observable Actions restriction to outcome Q and stopping time $T^{i,Q}$, we get that

$$V^*(\sigma,Q) \leqslant E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[r\int_0^{T^{i,Q}} e^{-rs}\big(g_i(A_s^{i,Q,\sigma},A_s^{-i,Q})ds - d\Gamma_s^{i,Q} + k\,d\Gamma_s^{-i,Q}\big) - r\Gamma_0^{i,Q} + rk\Gamma_0^{-i,Q}\Big] + \\ + E^{\mathbf{P}(A^{i,Q,\sigma},A^{-i,Q})}\Big[e^{-rT^{i,Q}}W_{T^{i,Q}}^{i,Q}\Big].$$

But the RHS of the above inequality is the value evaluated at the beginning of Q of the strategy that prescribes to follow σ until the moment of the first observable deviation and then instead of announcing this deviation, to abide to Q. By the base of induction, this value is weakly below $W^{i,Q}$. Thus, $V^*(\sigma, Q) \leqslant W^{i,Q}$.

`Necessity of 1.`

Suppose, $\mathcal{E}$ fails the One-Stage Deviation in Hidden Actions restriction. Then a profitable deviation σ can be constructed by deviating only in hidden actions within a given outcome Q, similarly to how it can be done in Sannikov (2007). Moreover, the value of this deviating strategy can be computed with the usual integrals so that $V^*(\sigma, Q) = V_*(\sigma, Q) > W^{i,Q}$.

`Necessity of 2.`

Suppose, $\mathcal{E}$ fails the One-Stage Deviation in Observable Actions restriction. Suppose the restriction fails for some outcome $Q \in \mathcal{E}$, Player i, and stopping time T. Consider the function $f(\omega) = e^{-rT(\omega)}\left(W^{i,\tilde{Q}(T)} + r\Delta\Gamma_T^{i,Q} - rk\Delta\Gamma_T^{-i,Q} - W_T^{i,Q}\Delta\Gamma_T^{-i,Q}\right)$, the discounted instantaneous gains from observably deviating at time $T(\omega)$. Then, $\{\omega \in \Omega^Q : f(\omega) > 0\}$ is not $(\mathcal{F}_T, \mathbf{P}^Q)$-measure zero set. Then $\exists\delta_1 > 0$, $\exists\delta_2 > 0$ such that the set $B(\delta_1) = \{\omega \in \Omega^Q : f(\omega) > \delta_1\}$ has $\mathbf{P}^Q$-outer measure relative to $\mathcal{F}_T$ equal to δ_2. As the stage-games payoffs are bounded and all money-transfer processes in $\mathcal{E}$ are uniformly non-manipulable, there exists a lower bound K such that $W^{i,\tilde{Q}(T)} \geqslant K$ on $\{\omega \in \Omega^Q : T(\omega) < \infty\}$. Take then a set $C \subseteq \{\omega : T(\omega) < \infty\}$ such that C is $\mathcal{F}_T$-measurable, $B(\delta_1) \subset C$, and $\mathbf{P}^Q(C) < \delta_2 + \frac{\delta_1\delta_2}{|K|}$. Consider the strategy σ for Player i prescribing no deviations in hidden actions and just one deviation in observable actions from outcome Q at $\hat{T} = T \cdot \mathbb{1}_C + \infty \cdot \mathbb{1}_{\Omega^Q \setminus C}$. This value of this strategy evaluated at the beginning of Q is at least $V^*(\sigma, Q) > W^{i,Q} + \delta_1\delta_2 + K \cdot \frac{\delta_1\delta_2}{|K|} \geqslant W^{i,Q}$. Thus, σ is a profitable deviation. The first part is proven.

Suppose further that $\mathcal{E}$ is measurable. Suppose the restriction fails for some outcome $Q \in \mathcal{E}$, Player i and stopping time T. Consider the set $B = \{\omega \in \Omega^Q : W_T^{i,Q} < W^{i,\tilde{Q}(T)} + r\Delta\Gamma_T^{i,Q} - rk\Delta\Gamma_T^{-i,Q}\}$. By measurability of $\mathcal{E}$, B is an $\mathcal{F}_T^Q$-measurable event. By the failure of the One-Stage Deviation in Observable Actions restriction, $Pr^{\mathbf{P}^Q}(B) > 0$. Define the stopping time $\hat{T} = T \cdot \mathbb{1}_B + \infty \cdot \mathbb{1}_{\Omega^Q \setminus B}$. Consider the strategy σ for Player i that prescribes no deviations in hidden actions and just one observable deviation from Q at $\hat{T}$. Clearly, this strategy will be a profitable deviation with $V^*(\sigma, Q) = V_*(\sigma, Q) > W^{i,Q}$.

B Proof of Theorem 2

B.1 Proof of Lemma 1

Take any point $(w_1, w_2) \in K(\epsilon_1)$. This point can be achieved as the expected payoff in some self-enforcing agreement $\mathcal{E}$ with inertia parameter ϵ_1. But then, $\mathcal{E}$ is also a self-enforcing agreement with inertia parameter ϵ_2 because it satisfies the conditions of Theorem 1 and because the ϵ_2-inertia restriction is less restrictive than the ϵ_1-inertia for $\epsilon_2 < \epsilon_1$. Thus, $(w_1, w_2) \in K(\epsilon_2)$.

B.2 Proof of Lemma 2

Clearly, the second statement in the formulation of Lemma 2 implies the first one. So it is sufficient to show that the second statement is correct. Suppose on the contrary that there is a public outcome Q in a self-enforcing agreement $\mathcal{E}$, a stopping time τ in Q, and Player i such that $W_\tau^{i,Q} \geqslant \underline{v}_i + r\Delta\Gamma_\tau^{i,Q} - rk\Delta\Gamma_\tau^{-i,Q}$ is violated on an event $A \in \mathcal{F}_\tau^Q$ of positive probability. Consider the following deviating strategy for Player i: follow the plan of actions and transfers suggested in Q on the event "not A"; on the event A, follow the proposed plans until τ and then switch to "dropping out from the cooperation," i.e., start always playing a hidden action that is a myopic best response against the current action of the opponent and always announce to refuse to send positive transfers if asked by the agreement. Notice, that the switch to the dropping-out regime happens only after time τ, at which point Player i will know whether A has happened or not. Thus, so described strategy for Player i is indeed a well-defined public strategy. Yet, this strategy will be a strictly profitable deviation, which contradicts the assumption that $\mathcal{E}$ is self-enforcing. Q.E.D.

B.3 Concatenation of Outcomes

In this subsection, I show how having two public outcomes Q^α and Q^β and a stopping time τ^α in outcome Q^α, one can construct the concatenated outcome $Con(Q^\alpha, Q^\beta, \tau^\alpha)$ which corresponds to the play of Q^A in the beginning until the time hits τ^α and then switches to the beginning of Q^B.

Suppose we are given two outcomes $Q^\alpha = \{\mathcal{P}^{Q^\alpha}, A^{Q^\alpha}, \Gamma^{Q^\alpha}\}$ and $Q^\beta = \{\mathcal{P}^{Q^\beta}, A^{Q^\beta}, \Gamma^{Q^\beta}\}$. Suppose τ^α is a stopping time in $\mathcal{P}^{Q^\alpha}$ at which the play should switch from Q^α to Q^β. Let us construct the concatenated outcome $Con(Q^\alpha, Q^\beta, \tau^\alpha) = \{\mathcal{P}, \mathcal{A}, \Gamma\}$:

- The state-space Ω for the concatenated outcome is the direct product of the state-spaces of Q^α and Q^β, i.e., $\Omega = \{\omega = (\omega_1, \omega_2) : \omega_1 \in \Omega^{Q^\alpha}, \omega_2 \in \Omega^{Q^\beta}\}$.
- The probability measure $\mathbf{P}$ for the concatenated outcome is the direct product $\mathbf{P} = \mathbf{P}^{Q^\alpha} \otimes \mathbf{P}^{Q^\beta}$.
- The moment of switch τ corresponds to τ^α, i.e., $\tau(\omega_1, \omega_2) = \tau^\alpha(\omega_1)$.
- The public filtration $(\mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0})$ is the following,
  - $\mathcal{F} = \sigma(\mathcal{F}^{Q^\alpha} \otimes \mathcal{F}^{Q^\beta})$;
  - $\mathcal{F}_t$ consists of all those events $A \in \mathcal{F}$ such that for any $0 \leqslant s_1 \leqslant s_2 \leqslant t$, the event $A \cap \{s_1 \leqslant \tau \leqslant s_2\}$ belongs to the σ-algebra $\sigma(\mathcal{F}^{Q^\alpha}_{s_2} \otimes \mathcal{F}^{Q^\beta}_{t-s_1})$ and the event $A \cap \{\tau > t\}$ belongs to $\mathcal{F}^{Q^\alpha}_t \otimes \{\varnothing, \Omega^{Q^\beta}\}$.
- The public signal X_t is $X_t(\omega_1, \omega_2) = X^{Q^\alpha}_t(\omega_1) \cdot \mathbb{1}_{\tau \geqslant t} + \Big(X^{Q^\alpha}_\tau(\omega_1) + X^{Q^\beta}_{t-\tau}(\omega_2)\Big) \cdot \mathbb{1}_{\tau < t}$.
- The recommended hidden actions A_t are $A_t(\omega_1, \omega_2) = A^{Q^\alpha}_t(\omega_1) \cdot \mathbb{1}_{\tau < t} + A^{Q^\beta}_{t-\tau}(\omega_2) \cdot \mathbb{1}_{\tau \geqslant t}$.
- The recommended cumulative money transfers Γ_t are $\Gamma_t = \Gamma^{Q^\alpha}_t(\omega_1) \cdot \mathbb{1}_{\tau < t} + \Big(\Gamma^{Q^\alpha}_\tau(\omega_1) + \Gamma^{Q^\beta}_{t-\tau}(\omega_2)\Big) \cdot \mathbb{1}_{\tau \geqslant t}$. Note that the switch happens only after the transfers recommended at time τ^α in Q^α are processed.

In this construction, $X_t - \int_0^t \mu(A_s) ds$ is a d-dimensional Brownian motion under $\mathbf{P}$, the processes A_t are progressively measurable for $(\mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0})$, and Γ_t are weakly-increasing nonnegative RCLL-processes adapted to $(\mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0})$. Thus, $Con(Q^\alpha, Q^\beta, \tau^\alpha)$ is a public outcome.

B.4 Proof of Theorem 2

By Assumption 1, there exists a self-enforcing agreement $\mathcal{E}$ delivering payoffs (w_1, w_2) with $w_1 > \underline{v}_1$ and $w_2 > \underline{v}_2$. Let Q^0 be the initial outcome of $\mathcal{E}$. By Lemma 2, we can find a modification for the processes of hidden actions (A^1_t, A^2_t) and the processes of promised continuation values (W^{1,Q^0}_t, W^{2,Q^0}) such that the One-Stage Deviation in Hidden Actions and the restriction $\forall i = 1, 2, W^{i,Q^0}_t \geqslant \underline{v}_i + r\Delta\Gamma^{i,Q^0}_t - rk\Delta\Gamma^{-i,Q^0}_t$ are satisfied always.

I will now construct the required Q^1 and Q^2.

Let us construct Q^1, the harshest punishment for Player 1:

Suppose first that the minmaxing profile $a = (a_1, a_2)$ for Player 1 is a Nash Equilibrium of the stage game. Then set Q^1 to be the public outcome corresponding to the play of (a_1, a_2) forever with zero volatility of promised continuation values, $\big(W^{1,Q^1}_t, W^{2,Q^1}_t\big) = \big(g_1(a), g_2(a)\big)$.

Suppose now that the profile $a = (a_1, a_2)$ minmaxing Player 1 is not a Nash equilibrium, but only enforceable. Then the construction is the following. Set $L_1 = (\underline{v}_1, w_2 + k(w_1 - \underline{v}_1))$ and

$L_2 = \big(\underline{v}_1, \underline{v}_2 + k(w_1 - \underline{v}_1) + k^2(w_2 - \underline{v}_2)\big)$. As $0 \leqslant k < 1$, L_1 is strictly above L_2. For the beginning of the outcome, call it Q, set $(W_0^1, W_0^2) = \frac{L_1+L_2}{2}$. Recommend the players to play the minmaxing profile (a_1, a_2) enforcing it along the vector $(0,1)$ and requiring no money transfers. I.e., set the matrix $B = (0,1)^\top \phi(a,(0,1))$. Construct (W_t^1, W_t^2) as a continuous weak solution to the system of SDE's

$$dW_t^i = r(W_t^i - g_i(a))dt + B^i(dX_t - \mu(a)dt).$$

Notice that for this solution, we will always have constant $W_t^1 = \underline{v}_1$. Thus, the solution (W_t^1, W_t^2) moves along the minmax line of Player 1. Consider the stopping time τ, the first time when (W_t^1, W_t^2) hits either L_1 or L_2. At L_1, require that the players send transfers $(w_1 - \underline{v}_1, 0)$, at L_2, require that they send transfers $\big(w_1 - \underline{v}_1 + k(w_2 - \underline{v}_2), w_2 - \underline{v}_2\big)$. Then stop the outcome Q and start playing the outcome Q^0. Define Q^1 as the concatenated outcome, $Q^1 = Con(Q, Q^0, \tau)$. Set the process of promised continuation values in the concatenated outcome as the concatenation of the processes of promised continuation values from Q and Q^0. Clearly, these processes will satisfy representation (1) and so they are indeed the processes of promised continuation values for Q^1. Moreover, Q^1 satisfies the One-Stage Deviation in Hidden Action restriction. Indeed, in the beginning, it is satisfied by enforceability of (a_1, a_2), and after the switch, it is satisfied for Q^0. Yet, Q^1 delivers the worst possible payoff to Player 1, $W^{1,Q^1} = \underline{v}_1$.

The outcome Q^2 for punishing Player 2 is constructed analogously.

Next, Q^1 does not require any transfers until the hitting time τ. As the incentives until τ are enforced by the constant matrix of volatilities, there exists $\epsilon_1 > 0$ such that Q^1 is a punishment outcome for any inertia parameter $\epsilon \in (0, \epsilon_1)$. Similarly, there exists $\epsilon_2 > 0$ such that Q^2 is a punishment outcome for any inertia parameter $\epsilon \in (0, \epsilon_2)$. Set the required $\bar{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$.

Finally, the agreements $\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$ satisfy the One-Stage Deviation in Observable Action restriction. Indeed, by construction, the processes of continuation values plus the current transfers always stay above the minmax payoffs $(\underline{v}_1, \underline{v}_2)$, exactly the payoffs promised to the players in case either of them deviates. Therefore, $\mathcal{E}^1(Q^1, Q^2)$ and $\mathcal{E}^2(Q^1, Q^2)$ are self-enforcing. Q.E.D.

C Proof of Theorem 5

C.1 Proof of Lemma 7

The proof is similar to the proof of Proposition 5 from Sannikov (2007). Indeed, notice first that the following adaptation of Proposition 3 from Sannikov (2007) applies to our case.

Proposition 3′. *Suppose that a curve $\mathcal{C}$ satisfies the optimality equation (4). Suppose further that $\mathcal{C}$ has endpoints which are attainable as payoffs in self-enforcing agreements with inertia parameter ϵ. Then any point in $\mathcal{C}$ is attainable as the payoff of a self-enforcing public agreement with inertia parameter ϵ, i.e., $\mathcal{C} \subseteq K(\epsilon)$.*

Proof. The construction in the proof is similar to the one used in the proof of Theorem 2. If the curve $\mathcal{C}$ has positive curvature, then the idea is to take any point $w \in \mathcal{C}$ and to construct the beginning of the outcome by supporting incentives without money-transfers, solely by the drift-diffusion of the promised continuation values along $\mathcal{C}$ until the values hit either of the endpoints of $\mathcal{C}$ (exactly how it is done in Sannikov (2007)). After that, use the concatenation with the initial outcomes of the corresponding self-enforcing agreements. Theorem 1 then will insure that so constructed outcome will be supportable in a self-enforcing agreement and will deliver to the players payoffs w. If $\mathcal{C}$ is a segment of a straight line, then w can be obtained as a public randomization between the agreements corresponding to the end points of $\mathcal{C}$. And so w will indeed be in $K(\epsilon)$. □

Then, notice that the following variant of Lemma 8 from Sannikov (2007) is valid in the current setting:

Lemma 8′. *Consider a point $w \in \partial_+ K(\epsilon) \backslash \mathcal{N}$ with the outward normal vector $\mathbf{N}$. Then the curve $\mathcal{C}$, which solves the optimality equation (4) with initial conditions $(w, \mathbf{N})$ does not enter the interior of $K(\epsilon)$.*

Proof. The proof uses Proposition 3′ and is otherwise the same as the proof of Lemma 8 in Sannikov (2007). □

Thus, indeed, the curvature of $\partial_+ K(\epsilon)$ can not be smaller than the one prescribed by the optimality equation (4). To prove that the curvature of $\partial_+ K(\epsilon)$ can not be greater than the one in the optimality equation (4), we use the following adaptation of Lemma 6′ from Hashimoto (2010).

Lemma 6″. *It is impossible for a solution $\mathcal{C}'$ of the optimality equation (4) with endpoints v_L and v_H to satisfy the following properties simultaneously:*

1. *There is a unit vector $\hat{\mathbf{N}}$ such that $\forall x > 0$, $v_L + x\hat{\mathbf{N}} \notin K(\epsilon)$ and $v_H + x\hat{\mathbf{N}} \notin K(\epsilon)$.*
2. *For all $w \in \mathcal{C}'$ with an outward unit normal $\mathbf{N}(w)$ for $\mathcal{C}'$ at w, we have*
$$\max_{v_N \in \mathcal{N}} \mathbf{N}(w) v_N < \mathbf{N}(w) w.$$
3. *$\mathcal{C}'$ "cuts through" $K(\epsilon)$, that is, there exists a point $v \in \mathcal{C}'$ such that $W_0 = v + x\hat{\mathbf{N}} \in K(\epsilon)$ for some $x > 0$.*
4. *$\inf_{w \in \mathcal{C}'} \hat{\mathbf{N}} \mathbf{N}(w)^\top > 0$, where $\mathbf{N}(w)$ is the outward normal vector for $\mathcal{C}'$ at w.*
5. *$\hat{\mathbf{N}}$ is positively correlated with the money-transfer vectors, $\hat{\mathbf{N}} \cdot (1, -k) \geqslant 0$ and $\hat{\mathbf{N}} \cdot (-k, 1) \geqslant 0$.*

Proof. The proof almost exactly repeats the proof from Hashimoto (2010). The only difference now is that with money transfers, the RHS of the Ito formula in footnote 2 of Hashimoto (2010) will have an extra term, $P_t = \int_0^t (1, -k) \cdot \hat{\mathbf{N}}\ d\Gamma_s^1 + \int_0^t (-k, 1) \cdot \hat{\mathbf{N}}\ d\Gamma_s^2 + (1, -k) \cdot \hat{\mathbf{N}}\ \Delta\Gamma_0^1 + (-k, 1) \cdot \hat{\mathbf{N}}\ \Delta\Gamma_0^2$. But since $\hat{\mathbf{N}}$ is positively correlated with both $(1, -k)$ and $(-k, 1)$, the term P_t is nonnegative. Therefore, equation (6) from Hashimoto (2010) still applies in our case and the rest of his proof works. □

To finish the proof of Lemma 7, take ϵ small enough that an optimal penal code exists. Take any point $w \in \partial_+ K(\epsilon) \backslash \mathcal{N}$. Set $\hat{\mathbf{N}}$ to be any outward unit-normal vector for $\partial_+ K(\epsilon)$ at w. By Lemma 3, the set $K(\epsilon)$ is comprehensive, and so $\hat{\mathbf{N}}$ is positively correlated with both $(1, -k)$ and $(-k, 1)$. If the curvature of $\partial_+ K(\epsilon)$ at w is greater than the one prescribed by the optimality equation or if $\partial_+ K(\epsilon)$ has a kink at w, then apply Lemma 6″ for w, $\hat{\mathbf{N}}$ and a solution $\mathcal{C}'$ which starts inside of $K(\epsilon)$ with the initial normal $\hat{\mathbf{N}}$ very close to w. This will lead to a contradiction. Therefore, the curvature of $\partial_+ K(\epsilon)$ at w must indeed be given by the optimality equation.

Finally, suppose $\partial_+ K(\epsilon)$ enters the minmax line for Player 1 at a point w outside of $\mathcal{N}$. We need to show then that $\partial_+ K(\epsilon)$ is tangent to $(1, -k)$ at w. Indeed, as $K(\epsilon)$ is comprehensive, the slope of $\partial_+ K(\epsilon)$ at w must be at least as steep as $-\frac{1}{k}$. But if that slop is even steeper, then we can apply Lemma 6″ for w, $\hat{\mathbf{N}} = (\frac{k}{\sqrt{1+k^2}}, \frac{1}{\sqrt{1+k^2}})$, and a solution starting inside $K(\epsilon)$ in the vicinity of w, which would yield a contradiction. Similarly, $\partial_+ K(\epsilon)$ must enter the minmax line for Player 2 either at a point from $\mathcal{N}$ or tangent to $(-k, 1)$. To finish the proof, it remains to notice that any point from $\mathcal{N}$ that is also an extreme point of $K(\epsilon)$ must correspond to the payoffs of some p-NE. Q.E.D.

C.2 Proof of Theorem 5

By Lemmata 3, 4, 5, 6, and 7, we already know that the set $K(\epsilon)$ must satisfy properties 1 and 2 from Theorem 5. It remains to show the converse, if K is a bounded set satisfying properties 1 and 2, then $cl(K) \subseteq K(\epsilon)$.

Indeed, take any $w \in \partial_+ K$. Let us construct an outcome Q^0 that will satisfy the One-Stage deviation restriction in Hidden Actions and deliver to the players the payoffs equal to w. There could be three different cases.

Case 1: $w \in \mathcal{N}$. Then take Q^0 as the initial public randomization among p-NE's of the stage game that would yield w followed by the infinite repetition of the corresponding realized p-NE without money transfers.

Case 2: $w \in \partial_+ K \backslash \mathcal{N}$ and the curvature of $\partial_+ K$ is strictly positive at w. Then start Q^0 as a weak solution to representation (1) with $W_0 = w$ that moves along the curve $\mathcal{C}$, which is the solution to the optimality equation (4) with the initial condition $(w, \mathbf{N}(w))$. The underlying action profile is going to be determined as the maximizer in the optimality equation. As the volatility along $\mathcal{C} \backslash \mathcal{N}$ is uniformly bounded away from 0, the curve $\mathcal{C}$ eventually hits either a payoff from $\mathcal{N}$ or the minmax of either of the players. In the former case, concatenate the play with the subsequent randomization and indefinite play of the realized p-NE. In the later, when $\mathcal{C}$ hits the minmax line of Player i at point v, introduce money transfers from Player i made with the retention parameter k such that they coincide with the pushing process of W_t on $\mathcal{C}$ with the reflection boundary at v. So constructed money-transfer processes will be M-nonmanipulable for some $M > 0$. Indeed, if the reflexion happens only on one end of $\mathcal{C}$, then the rate of growth of the transfers as time $t \to \infty$ is that of order $\sqrt{t}$. If the reflexion happens on both ends, the rate of growth is of order t. As there are only finitely many hidden action profiles and as the volatility of W_t is uniformly bounded on $\mathcal{C}$, there will exist a constant $C > 0$, such that for any $t > 0$, any manipulations with the drift of the public signal can not increase either of the cumulative transfers by more than Ct. Since the interest rate $r > 0$, the money-transfers processes indeed will be nonmanipulable. Thus, Q^0 will be the required public outcome. Support Q^0 by the optimal penal code from Theorem 2. This will give us a self-enforcing agreement with payoffs w.

Case 3: $w \in \partial_+ K \backslash \mathcal{N}$, but the curvature of $\partial_+ K$ at w is 0. Then the solution to the optimality equation with the initial condition $(w, \mathbf{N}(w))$ is a straight line. As $K(\epsilon)$ is bounded, this solution has to stop somewhere. If both of the endpoints are in $\mathcal{N}$, then w can be obtained by initial public randomization between the agreements corresponding to this two endpoints. If one of the endpoints is on minmax line of Player i, while another is in $\mathcal{N}$, then w can be obtained in the agreement which first asks Player i to transfer positive amount of money to jump to the endpoint in $\mathcal{N}$, and then follows with the agreement corresponding to this end point. Finally, as $k \neq 1$, it is not possible for a straight solution $\mathcal{C}$ to enter both minmax lines while at the same time being parallel to $(1, -k)$ and $(-k, 1)$.

Now, if w is an end point of $\partial_+ K$ which is not in $\mathcal{N}$, we can construct an agreement by using the money transfers to push away from the minmax lines similarly to how it is done in Case 2.

Finally, any point that may be obtained from $cl(\partial_+ K)$ by subtracting the money-transfer vectors will also belong to $K(\epsilon)$ by Lemma 3.

Thus, $cl(K) \subseteq K(\epsilon)$. Q.E.D.